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A Banach contraction principle in fuzzy metric spaces defined by means of t-conorms

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Abstract Fixed point theory in fuzzy metric spaces has grown to become an intensive field of research. The difficulty of demonstrating a fixed point theorem in such kind of spaces makes the authors to demand extra conditions on the space other than completeness. In this paper, we introduce a new version of the celebrated Banach contracion principle in the context of fuzzy metric spaces. It is defined by means of t-conorms and constitutes an adaptation to the fuzzy context of the mentioned contracion principle more "faithful" than the ones already defined in the literature. In addition, such a notion allows us to prove a fixed point theorem without requiring any additional condition on the space apart from completeness. Our main result (Theorem 1) generalizes another one proved by Castro-Company and Tirado. Besides, the celebrated Banach fixed point theorem is obtained as a corollary of Theorem 1.

Keywords Fuzzy metric space · Fuzzy contractive mapping · Archimedean continuous t-conorm · Fixed point · k-contraction

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1 Introduction

The issue of providing a fuzzy version of the concept of classical metric became a field of interest in the second half of the last century. Kramosil and Michalek contributed to it by introducing in [18] a notion of fuzzy metric space by

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means of the so-called continuous t-norms. Such a notion actually constitutes an adaptation to the fuzzy context of probabilistic metric spaces due to Menger (see [19]). Nowadays, the fuzzy metrics defined by Kramosil and Michalek are commonly managed as the reformulation of them provided by Grabiec in [8]. Later on, with the aim of retrieving more faithfully the classical notion of metric to the fuzzy context, George and Veeramani modified in [3] some axioms of the ones established by the Grabiec's reformulation to introduce a new concept of fuzzy metric (we will refer to them as GV-fuzzy metrics). Moreover, in [3] it was shown that each GV-fuzzy metric induces a (crisp) topology. This fact can also be demonstrated for fuzzy metrics by attending to the results provided in [23]. Since then, many research works have been devoted to the study of both aforementioned concepts of fuzzy metrics (see for instance [4,5,10,9,12–14,24] or recent publications as [11,16,22,28]). Moreover, many results demonstrated to GV-fuzzy metrics can be retrieved for fuzzy metrics, and vice-versa. For instance, in [12] it was proved that GV-fuzzy metrics are metrizable, which can also be obtained for fuzzy metrics throughout the results given in [24]. Nevertheless, there exist differences between fuzzy metrics and GV-fuzzy metrics. For instance, GV-fuzzy metrics are non-completable, in general, (see [13,14]) in contrast to the case of the concept due to Kramosil and Michalek.

Coming back to what was aforesaid, fuzzy metric spaces and GV-fuzzy ones are metrizable. This means that both concepts are topologically equivalent to classical metrics. So, one can wonder whatever are new in these fuzzy versions of classical metrics. A topic which substantially differs from the classical one is the fixed point theory. Indeed, many researchers have tried to addapt some classical fixed point results to the fuzzy context (see, for instance, [1, 6–8,15,20,21,26,29]). Nevertheless, in such adaptations we usually find some inconveniences. For instance, how to define a contractive mapping in a fuzzy metric space has been approached in different ways. Besides, the completeness of the fuzzy metric is not usually enough to prove a fuzzy version of a classical fixed point theorem. So, some authors have opted to use a stronger version of completeness whereas other ones have chosen to demand extra conditions on the space in order to establish their fixed point theorems.

The aim of this paper is to provide a new version of the celebrated Banach fixed point theorem in classical metrics to the fuzzy setting. The significance of our approach to the fixed point theory in fuzzy metrics is twofold. On the one hand, the contractive condition used can be seen as a faithful adaptation of the classical one, in such a way that de fuzzy distance between the images of two elements is greater than the fuzzy distance between such elements "multiplied" by a constant $k \in]0,1[$ (see Definition 7). On the other hand, our main theorem does not demand any extra condition to the completeness of the fuzzy metric space, but a condition on the contraction defined. In addition, it is demonstrated by means of a counterexample that such a condition cannot be removed to obtain a fixed point. Moreover, we demonstrate that our contractive condition generalizes another one already appeared in [27]. Furthermore, our main theorem generalizes a fixed point theorem proved in [2]. Finally, the celebrated Banach fixed point theorem is obtained as a corollary of Theorem 1.

2 Preliminaries

We begin this section recalling the notion of t-norm, which was used to define the concept of fuzzy metric that we will manage. Our main reference for t-norms is [17].

Definition 1 A binary operation $*$ on $[0,1]$ is called a *t-norm* if, for each $a, b, c \in [0,1]$, the following four axioms are satisfied:

(T1) $a * b = b * a$;
(T2) $a * (b * c) = (a * b) * c$;
(T3) $a * b \leq a * c$ whenever $b \leq c$;
(T4) $a * 1 = a$.

If in addition, the function $* : [0,1]^2 \to [0,1]$ is continuous, we will say that $*$ is a continuous t-norm.

The most commonly used continuous t-norms in Fuzzy Logic are the *minimum* t-norm $*_M$, given by $a *_M b = \min\{a, b\}$ for each $a, b \in [0,1]$, the *product* t-norm $*_P$, given by $a *_P b = a \cdot b$ for each $a, b \in [0,1]$, and the *Lukasievicz* t-norm $*_L$, given by $a *_L b = \max\{a + b - 1, 0\}$ for each $a, b \in [0,1]$. Moreover, the largest t-norm is the minimum t-norm and, in addition, $*_M \geq *_P \geq *_L$.

A particular kind of t-norms are the so-called Archimedean t-norms, which are defined as follows.

Definition 2 A t-norm $*$ is said to be Archimedean if for each $a, b \in]0,1[$ there exists $n \in \mathbb{N}$ such that $a_*^{(n)} < b$, where $a_*^{(n)}$ denotes (throughout the paper) $a * \cdots *^{(n)} a$.

Examples of (continuous) Archimedean t-norms are $*_P$ and $*_L$.

An immediate consequence of Definition 2 is that each Archimedean t-norm satisfies the so-called limit property, i.e. for each $a \in]0,1[$ it is hold $\lim_n a_*^{(n)} = 0$. In addition, from such a property we deduce that $a * a < a$ for each $a \in]0,1[$.

For each t-norm we can find a dual operator of it that is known as t-conorm. Below we recall such a notion.

Definition 3 A binary operation $\diamond$ on $[0,1]$ is called a t-co*norm* if, for each $a, b, c \in [0,1]$, the following four axioms are satisfied:

(S1) $a * b = b * a$;
(S2) $a * (b * c) = (a * b) * c$;
(S3) $a * b \leq a * c$ whenever $b \leq c$;
(S4) $a \diamond 0 = a$.

If in addition, the function $\diamond : [0,1]^2 \to [0,1]$ is continuous, we will say that $\diamond$ is a continuous t-conorm.

The next proposition shows the duality relationship between t-norms and t-conorms.

Proposition 1 *A binary operation $\diamond$ on $[0,1]$ is a t-conorm if and only if there exists a t-norm $*$ such that, for each $a, b \in [0,1]$, it is satisfied the following*

$$a \diamond b = 1 - ((1-a) * (1-b)).$$

In such a case, we will say that $\diamond$ is the dual t-conorm of the t-norm $*$, or vice-versa.

The dual t-conorms of the previous examples of t-norms are, the *maximum* t-conorm $\diamond_M$, given by $a \diamond_M b = \max\{a, b\}$ for each $a, b \in [0,1]$, the *algebraic sum* t-conorm $\diamond_P$, given by $a \diamond_P b = a + b - a \cdot b$ for each $a, b \in [0,1]$, and the *bounded sum* t-conorm $\diamond_L$, given by $a \diamond_L b = \min\{a+b, 1\}$ for each $a, b \in [0,1]$, respectively.

The least t-conorm is the maximum t-conorm and, in addition, $\diamond_M \leq *_P \leq *_L$. So, given a t-conorm $\diamond$ we have that $a \diamond b \geq a$, for each $a, b \in [0,1]$.

In the case of t-conorms, the Archimedean ones are defined as follows.

Definition 4 A t-conorm $\diamond$ is said to be Archimedean if for each $a, b \in]0,1[$ there exists $n \in \mathbb{N}$ such that $a_\diamond^{(n)} > b$, where $a_\diamond^{(n)}$ denotes (throughout the paper) $a \diamond \cdots \diamond^{(n)} a$.

Similarly to the case of t-norms, it follows directly form the previous definition that Archimedean t-conorms satisfy the following property.

Proposition 2 *Let $\diamond$ be an Archimedean t-conorm. Then, $\lim_n a_\diamond^{(n)} = 1$ for each $a \in]0,1[$. Besides, $a \diamond a > a$ for each $a \in]0,1[$.*

Now, we are able to recall the reformulation presented by Grabiec in [8] of the notion of fuzzy metric introduced by Kramosil and Michalek introduced in [18].

Definition 5 A *fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$;
(KM5) The assignment $M(x, y, \_) : [0, \infty[\to [0,1]$ is a left-continuous function.

In such a case $(M, *)$, or simply M, is called a fuzzy metric on X.

It is well known that each fuzzy metric M on X induces a topology $\mathcal{T}_M$ on X which has as a base the following family of open balls

$$\mathcal{B} = \{B_M(x,r,t) : x \in X, r \in]0,1[, t > 0[\},$$

where $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$ for each $x \in X$, $r \in]0,1[$ and $t > 0$. Moreover, convergent sequences in fuzzy metric spaces are characterized as follows.

Proposition 3 *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ in $(X, \mathcal{T}_M)$ if and only if $\lim_n M(x_n, x, t) = 1$ for each $t > 0$, i.e. for each $\epsilon \in]0,1[$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \epsilon$ for each $n \geq n_0$.*

Finally, we recall the notion of Cauchy sequence and completeness in the context of fuzzy metric spaces (see [3,25]).

Definition 6 A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space (X, M) is said to be *Cauchy* if $\lim_{n,m} M(x_n, x_m, t) = 1$, i.e. if for each $\epsilon \in]0,1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$.

As usual, $(X, M, *)$ is called *complete* if every Cauchy sequence in X is convergent with respect to $\mathcal{T}_M$.

3 The results

We begin this section by introducing the next notion of fuzzy contractive mapping.

Definition 7 Let $(X, M, *)$ be a fuzzy metric space. We will say that a mapping $T : X \to X$ is a *fuzzy k-$\diamond$-contraction* if there exists $k \in]0,1[$ and a continuous t-conorm $\diamond$ satisfying, for each $x, y \in X$ and $t > 0$, the following condition:

$$M(T(x), T(y), t) \geq k \diamond M(x, y, t). \tag{1}$$

To illustrate the previous definition we present the following example. It provides fuzzy k-$\diamond$-contractions for the most commonly Archimedean t-conorms used in Fuzzy Logic.

Example 1 Let $(X, M, *)$ be a fuzzy metric space and let $T : X \to X$ be a mapping.

T is a fuzzy k-$\diamond_L$-contraction if there exists $k \in]0,1[$ satisfying, for each $x, y \in X$ and $t > 0$, the following condition:

$$M(T(x), T(y), t) \geq \min\{k + M(x, y, t), 1\}. \tag{2}$$

T is a fuzzy k-$\diamond_P$-contraction if there exists $k \in]0,1[$ satisfying, for each $x, y \in X$ and $t > 0$, the following condition:

$$M(T(x), T(y), t) \geq k + M(x, y, t) - k \cdot M(x, y, t). \tag{3}$$

T is a fuzzy k-$\diamond_M$-contraction if there exists $k \in]0,1[$ satisfying, for each $x, y \in X$ and $t > 0$, the following condition:

$$M(T(x), T(y), t) \geq \max\{k, M(x, y, t)\}. \quad (4)$$

Obviously, if $\diamond_1$ and $\diamond_2$ are t-conorms, such that $\diamond_1 \leq \diamond_2$, then each fuzzy k-$\diamond_2$-contraction is a fuzzy k-$\diamond_1$-contraction. So, each fuzzy k-$\diamond_L$-contraction is a fuzzy k-$\diamond_P$-contraction since $\diamond_P \leq \diamond_L$. Nevertheless, the reciprocal of such an affirmation is not true as shows the next example.

Example 2 Let $(X, M_1, *_L)$ be the fuzzy metric space, where $X = [0,1]$ and, for each $x, y \in [0,1]$, $M_1(x, y, t) = 1 - |x - y|$ for each $t > 0$, and $M_1(x, y, 0) = 0$.

Consider the mapping $T : [0,1] \to [0,1]$ given by $T(x) = \frac{x}{2}$, for each $x \in [0,1]$.

Then, T is a fuzzy $\frac{1}{2}$-$\diamond_P$-contraction. Indeed, for each $x, y \in X$ and $t > 0$, we have that

$$M_1(T(x), T(y), t) = 1 - \frac{1}{2}|x - y| = \frac{1}{2} + M_1(x, y, t) - \frac{1}{2} \cdot M_1(x, y, t).$$

However, for each $k \in]0,1[$, T is not a fuzzy k-$\diamond_L$-contraction as we show below.

Fix $k \in]0,1[$ and let $y \in]0,k]$. On the one hand, $M(T(0), T(y), t) = 1 - \frac{y}{2} < 1$. On the other hand, $k + M_1(0, y, t) = k + 1 - y \geq k + 1 - k = 1$. Then,

$$M_1(T(0), T(y), t) < k \diamond_L M_1(0, y, t).$$

So, taking into account that $k \in]0,1[$ is arbitrary, we conclude that T is not a k-$\diamond_L$-contraction.

We are now able to demonstrate the following fixed point result.

Theorem 1 *Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy k-$\diamond$-contraction. If $\diamond$ is Archimedean, then T has a unique fixed point.*

Proof Let $x \in X$ and define the sequence $\{x_n\}_{n \in \mathbb{N}}$ recursively as follows: $x_1 = T(x)$ and $x_n = T(x_{n-1})$, for each $n \geq 2$. We will show, by contradiction, that $\{x_n\}$ is a Cauchy sequence.

First of all, by hypothesis, there exist $k \in]0,1[$ and an Archimedean t-conorm satisfying, for each $x, y \in X$ and $t > 0$, the following condition:

$$M(T(x), T(y), t) \geq k \diamond M(x, y, t). \quad (5)$$

We claim that, for each $t > 0$, it is fulfilled $M(x_{n+1}, x_n, t) \geq k_\diamond^{(n)}$, for each $n \in \mathbb{N}$. We will prove such an affirmation by induction.

Fix $t > 0$. By (5) we have that $M(x_2, x_1, t) \geq k \diamond M(x_1, x, t) \geq k$. So, $M(x_2, x_1, t) \geq k$ and the case $n = 1$ is satisfied. Let $n \in \mathbb{N}$ and suppose that our affirmation is true for each $m \leq n$. We will see that it is also fulfilled for

$n+1$. Again, by (5) we have that $M(x_{n+2}, x_{n+1}, t) \geq k \diamond M(x_{n+1}, x_n, t)$. Furthermore, by induction hypothesis we have that $M(x_{n+1}, x_n, t) \geq k_\diamond^{(n)}$. Then, $M(x_{n+2}, x_{n+1}, t) \geq k \diamond k_\diamond^{(n)} = k_\diamond^{(n+1)}$ and so the case $n+1$ is hold. Moreover, since $t > 0$ is arbitrary, we conclude that, for each $t > 0$, $M(x_{n+1}, x_n, t) \geq k_\diamond^{(n)}$, for each $n \in \mathbb{N}$, as we claimed. Thus, for each $n \in \mathbb{N}$, we have that $\bigwedge_{t>0} M(x_{n+1}, x_n, t) \geq k_\diamond^{(n)}$. So, $\lim_n \left(\bigwedge_{t>0} M(x_{n+1}, x_n, t)\right) \geq \lim_n k_\diamond^{(n)} = 1$.

Now, assume that $\{x_n\}_{n\in\mathbb{N}}$ is not Cauchy. Then, there exists $\epsilon \in]0,1[$ and $t > 0$ such that, for each $n \in \mathbb{N}$ we can find $m(n) > l(n) \geq n$ satisfying $M(x_{m(n)}, x_{l(n)}, t) \leq 1-\epsilon$. Under this assumption, we construct two subsequences $\{x_{m_n}\}_{n\in\mathbb{N}}$ and $\{x_{l_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, as follows.

Let $n = 1$. We take $l_1 = l(1)$ and let m_1 the least integer greater than $l(1)$ satisfying $M(x_{m_1}, x_{l_1}, t) \leq 1-\epsilon$, i.e. for such elements we have that $M(x_{m_1} - 1, x_{l_1}, t) > 1-\epsilon$. The subsequent elements of both subsequences are picked recursively as follows. For each $n \geq 1$, consider $m_n \in \mathbb{N}$. Then, there exist $m(m_n) > l(m_n) \geq m_n (> l_n)$ such that $M(x_{m(m_n)}, x_{l(m_n)}, t) \leq 1-\epsilon$. We take $l_{n+1} = l(m_n)$ and let m_{n+1} the least integer greater than $l(m_n)$ satisfying $M(x_{m_{n+1}}, x_{l_{n+1}}, t) \leq 1-\epsilon$.

Then, for each $n \in \mathbb{N}$ and each $s \in]0,t[$, we have that

$$1-\epsilon \geq M(x_{m_n}, x_{l_n}, t) \geq M(x_{m_n}, x_{m_n-1}, s) * M(x_{m_n-1}, x_{l_n}, t-s) \geq$$

$$\geq \left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n-1}, t)\right) * M(x_{m_n-1}, x_{l_n}, t-s).$$

Therefore, since the function $M(x, y, \_)$ is left-continuous, for each $x, y \in X$, using the previous inequalities we obtain, for each $n \in \mathbb{N}$, the next inequalities

$$1-\epsilon \geq M(x_{m_n}, x_{l_n}, t) \geq \left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n-1}, t)\right) * M(x_{m_n-1}, x_{l_n}, t) \geq$$

$$\geq \left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n-1}, t)\right) * (1-\epsilon).$$

So, taking limit as n tends to ∞ in the preceding inequality we deduce, by the continuity of $*$, that

$$1-\epsilon \geq \lim_n M(x_{m_n}, x_{l_n}, t) \geq \lim_n \left(\left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n-1}, t)\right) * (1-\epsilon)\right) =$$

$$= \left(\lim_n \left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n-1}, t)\right)\right) * \left(\lim_n (1-\epsilon)\right) \geq$$

$$\geq \left(\lim_n k_\diamond^{(n)}\right) * (1-\epsilon) = 1 * (1-\epsilon) = 1-\epsilon.$$

Thus, we conclude that $\lim_n M(x_{m_n}, x_{l_n}, t) = 1-\epsilon$.

On the other hand, by the contractive condition, we have, for each $n \in \mathbb{N}$ and each $s \in]0,t[$,

$$M(x_{m_n}, x_{l_n}, t) \geq M(x_{m_n}, x_{m_n+1}, s/2) * M(x_{m_n+1}, x_{l_n+1}, t-s) * M(x_{l_n+1}, x_{l_n}, s/2) \geq$$

$$\geq M(x_{m_n}, x_{m_n+1}, s/2) * (k \diamond M(x_{m_n}, x_{l_n}, t-s)) * M(x_{l_n+1}, x_{l_n}, s/2) \geq$$

$$\geq \left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n+1}, t) \right) * (k \diamond M(x_{m_n}, x_{l_n}, t-s)) * \left(\bigwedge_{t>0} M(x_{l_n+1}, x_{m_l}, t) \right)$$

Then, using the same arguments used above we obtain, for each $n \in \mathbb{N}$

$$M(x_{m_n}, x_{l_n}, t) \geq$$

$$\geq \left(\bigwedge_{t>0} M(x_{m_n}, x_{m_n+1}, t) \right) * (k \diamond M(x_{m_n}, x_{l_n}, t)) * \left(\bigwedge_{t>0} M(x_{l_n+1}, x_{m_l}, t) \right).$$

Again, taking limit as n tends to ∞, the continuity of $*$ and $\diamond$ ensure

$$1 - \epsilon = \lim_n M(x_{m_n}, x_{l_n}, t) \geq k \diamond \left(\lim_n M(x_{m_n}, x_{l_n}, t) \right) = k \diamond (1 - \epsilon).$$

The fact that $\diamond$ is Archimedean provides the contradiction, since $k \diamond (1-\epsilon) > 1 - \epsilon$.

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and, since $(X, M, *)$ is complete there exists $x \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x, i.e. $\lim_n M(x_n, x, t) = 1$ for each $t > 0$. We will see that x is a fixed point of T.

Fix $t > 0$, then, for each $n \in \mathbb{N}$, we have that

$$M(x, T(x), t) \geq M(x, x_n, t/2) * M(x_n, T(x), t/2)$$

$$\geq M(x, x_n, t/2) * (k \diamond M(x_{n-1}, x, t/2)).$$

Taking limits as n tends to ∞ we obtain, by continuity of $*$ and $\diamond$, the next

$$M(x, T(x), t) \geq \left(\lim_n M(x, x_n, t/2) \right) * \left(k \diamond \left(\lim_n M(x_{n-1}, x, t/2) \right) \right) = 1 * (k \diamond 1) = 1.$$

Thus, since $t > 0$ is arbitrary, we conclude that $M(x, T(x), t) = 1$ for each $t > 0$, or equivalently, $T(x) = x$.

Finally, it remains to prove the uniqueness of x. Suppose that $T(y) = y$ for some $y \in X$. Then, by the contractive condition we have that, for each $t > 0$,

$$M(x, y, t) = M(T(x), T(y), t) \geq k \diamond M(x, y, t).$$

So, since $\diamond$ is Archimedean we deduce that $M(x, y, t) = 1$, for each $t > 0$, which implies that $x = y$.

Observe in the preceding theorem that the condition of being Archimedean on the t-conorm is used to show that, for any arbitrary $x_0 \in X$, the iterative sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is Cauchy. The next example shows that such a condition cannot be removed to obtain the conclusion of the theorem.

Example 3 Consider the tern $(X, M, \wedge)$, where $X = \mathbb{R}$ and M is given, for each $x, y \in X$, by

$$M(x,y,t) = \begin{cases} \frac{1}{2}, & \text{if } t \le |x-y| \\ 1, & \text{if } t > |x-y| \end{cases},$$

for each $t > 0$, and $M(x, y, 0) = 0$. $(M, \wedge)$ is a fuzzy metric on X. Indeed, it is not hard to check that M satisfies axioms $(KM1)$, $(KM2)$, $(KM3)$ and $(KM5)$. So, we will see that M also fulfils $(KM4)$.

Let $x, y, z \in X$ and $t, s > 0$. The case $t+s > |x-z|$ implies $M(x, z, t+s) = 1$ and so the inequality holds. So, assume that $t+s \le |x-z|$. In such a case, we just can consider two possibilities:

i) Suppose that $t \le |x-y|$ and $s \le |y-z|$. Then,

$$M(x,z,t+s) = \frac{1}{2} \ge \frac{1}{2} \wedge \frac{1}{2} = M(x,y,t) \wedge M(y,z,s).$$

ii) Suppose that $t \le |x-y|$ and $s > |y-z|$ (or $t > |x-y|$ and $s \le |y-z|$). Then,

$$M(x,z,t+s) = \frac{1}{2} \ge \frac{1}{2} \wedge 1 = M(x,y,t) \wedge M(y,z,s).$$

Observe that, the case $t > |x-y|$ and $s > |y-z|$ implies $t+s > |x-y|+|y-z| \ge |x-z|$, which has been considered above.

Besides, $(X, M, \wedge)$ is complete. To show this fact, we will see first that each Cauchy sequence in $(X, M, \wedge)$ it is so in $\mathbb{R}$ endowed with the usual metric d_u.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, M, \wedge)$. Then, for each $\delta \in \left]0, \frac{1}{2}\right[$ there exists $n_\delta \in \mathbb{N}$ such that $M(x_n, x_m, \delta) > 1-\delta > 1-\frac{1}{2}$ for each $n, m \ge n_\delta$. So, by definition of M, $|x_n - x_m| < \delta$ for each $n, m \ge n_\delta$. Then, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, d_u)$ and, in consequence, $\{x_n\}_{n\in\mathbb{N}}$ converges to (some) $x \in \mathbb{R}$, in $(\mathbb{R}, d_u)$. Therefore, for each $\epsilon > 0$ we can find n_ϵ satisfying $|x_n - x| < \epsilon$ for each $n \ge n_\epsilon$. It remains to prove that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $(X, M, \wedge)$.

Let $\epsilon \in]0,1[$ and $t > 0$. Considering $\epsilon' = t > 0$, since $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $(\mathbb{R}, d_u)$, there exists n_0 such that $|x_n - x| < \epsilon' = t$ for each $n \ge n_0$. Then, by definition of M, we have that $M(x_n, x, t) = 1 > 1 - \epsilon$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $(X, M, \wedge)$ and, we conclude that $(X, M, \wedge)$ is a complete fuzzy metric space.

Define $T : X \to X$, given by $T(x) = x + 1$ for each $x \in X$. Obviously, T has not any fixed point. Furthermore, it is not hard to check that $M(T(x), T(y), t) = M(x, y, t)$, for each $x, y \in X$ and $t > 0$.

Now, consider the continuous t-conorm $\diamond_M$. It is well known that $\diamond_M$ is not Archimedean. Moreover, for each $k \in]0, \frac{1}{2}]$ we have that

$$M(T(x), T(y), t) = M(x, y, t) \ge k \diamond_M M(x, y, t).$$

Then, T is a fuzzy k-$\diamond_M$-contraction on a complete fuzzy metric space which has not fixed point.

We continue our study showing the significance of our main theorem by using it to generalize a fixed point theorem proved by Castro-Company and Tirado in [2]. Such a result demanded a restriction on the continuous t-norm that defines the fuzzy metric under consideration. The mentioned restriction involves a family of continuous t-norm known as Yager t-norms. We recall this family of t-norms below.

Given $\lambda \in]0,\infty[$, we will say that $*_Y^\lambda$ is a Yager t-norm if it is defined, for each $a,b \in [0,1]$, as follows

$$a *_Y^\lambda b = \max\left\{1-\left((1-a)^\lambda + (1-b)^\lambda\right)^{\frac{1}{\lambda}}, 0\right\}.$$

Now, we are able to recall the fixed point theorem aforementioned.

Theorem 2 *(Castro-Company and Tirado [2].) Let $(X,M,*)$ be a complete fuzzy metric space such that $* \geq *_Y^\lambda$, for some $\lambda \in]0,\infty[$, and let $T : X \to X$. If there exists $c \in]0,1[$ satisfying $M(T(x),T(y),t) \geq 1-c+cM(x,y,t)$, for each $x,y \in X$ and $t>0$, then T has a unique fixed point.*

In addition to demonstrate the previous theorem, the authors in [2] discussed if such a theorem involves each fuzzy metric, i.e. if for each continuous t-norm $*$ we can find $\lambda \in]0,\infty[$ satisfying $* \geq *_Y^\lambda$. In this direction, Castro-Company and Tirado proved that such an affirmation is not true, in general. Indeed, they provided an example of continuous t-norm for which does not exist any $\lambda \in]0,\infty[$ such that $* \geq *_Y^\lambda$.

So, Theorem 2 cannot be applied to an arbitrary complete fuzzy metric space. We will see that the condition on the t-norm posed in such a theorem can be removed to obtain the result. To this end, observe that the contractive condition used in Theorem 2 is a particular case of k-$\diamond$-contraction. Indeed, such a contraction is actually a k-$\diamond_P$-contraction when we consider $k = 1-c \in]0,1[$. In this case, on account of expression (3), the contractive condition turns as follows:

For each $x,y \in X$ and $t>0$ it is satisfied the next

$$M(T(x),T(y),t) \geq (1-c)+M(x,y,t)-(1-c)\cdot M(x,y,t) = 1-c+cM(x,y,t).$$

Hence, by Theorem 1 we obtain the next generalization of Theorem 2.

Corollary 1 *Let $(X,M,*)$ be a complete fuzzy metric space and let $T : X \to X$. If there exists $c \in]0,1[$ satisfying $M(T(x),T(y),t) \geq 1-c+cM(x,y,t)$, for each $x,y \in X$ and $t>0$, then T has a unique fixed point.*

We finish our work showing that the celebrated Banach fixed point theorem in classical metric spaces is a corollary of Theorem 1.

Corollary 2 *(Classical Banach fixed point theorem.) Let (X,d) be a complete metric space and let $T : X \to X$ be a contractive mapping, i.e. there exists $k \in]0,1[$ such that*

$$d(T(x),T(y)) \leq k \cdot d(x,y), \text{ for each } x,y \in X.$$

Then, T has a unique fixed point.

Proof Let (X,d) be a complete metric space and let $T: X \to X$ be a contractive mapping. Define, for each $x, y \in X$, $\tilde{M}_d(x,y,t) = \max\{1-d(x,y),0\}$ for each $t>0$, and $\tilde{M}_d(x,y,0)=0$.

It is not hard to check that $(X, \tilde{M}_d, *_L)$ is a complete fuzzy metric space. We are showing that T is a fuzzy k-$\diamond_P$-contraction for $c = 1-k \in]0,1[$. Let $x, y \in X$. We distinguish two possibilities:

1. Suppose $d(T(x),T(y)) \geq 1$. Then, $\tilde{M}_d(T(x),T(y),t)=0$ for each $t>0$. On the other hand, $d(x,y) \geq k \cdot d(x,y) \geq d(T(x),T(y)) \geq 1$. So, $\tilde{M}_d(x,y,t)=0$ for each $t>0$. Therefore,
$$\tilde{M}_d(T(x),T(y),t) = 0 = k \diamond_P \tilde{M}_d(x,y,t), \text{ for each } t>0.$$
2. Assume now $d(T(x),T(y)) < 1$. Then, $\tilde{M}_d(T(x),T(y),t) = 1-d(T(x),T(y))$ for each $t>0$. Moreover, $\tilde{M}_d(x,y,t) \leq 1-d(x,y)$ for each $t>0$. Therefore, for each $t>0$ we have that
$$\tilde{M}_d(T(x),T(y),t) = 1-d(T(x),T(y)) \geq 1-k\cdot d(x,y) = 1-k+k-k\cdot d(x,y) =$$
$$= 1-k+k\cdot(1-d(x,y)) = c+(1-c)\cdot(1-d(x,y)) =$$
$$= c+(1-d(x,y))-c\cdot(1-d(x,y)) = c \diamond_P (1-d(x,y)) \geq c \diamond_P \tilde{M}_d(x,y,t).$$

Thus, T is a fuzzy c-$\diamond_P$-contraction and, taking into account that $\diamond_P$ is Archimedean, by Theorem 1 we conclude that T has a unique fixed point.

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