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Additional Information

A continuum and computational framework for viscoelastodynamics: Finite deformation linear models

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Abstract

This work concerns the continuum basis and numerical formulation for deformable materials with viscous dissipative mechanisms. We derive a viscohyperelastic modeling framework based on fundamental thermomechanical principles. Since most large deformation problems exhibit isochoric properties, our modeling work is constructed based on the Gibbs free energy in order to develop a continuum theory using pressure-primitive variables, which is known to be well-behaved in the incompressible limit. A set of general evolution equations for the internal state variables is derived. With that, we focus on a family of free energies that leads to the so-called finite deformation linear model. Our derivation elucidates the origin of the evolution equations of that model, which was originally proposed heuristically and thus lacked formal compatibility with the underlying thermodynamics. In our derivation, the thermodynamic inconsistency is clarified and rectified. A classical model based on the identical polymer chain assumption is revisited and is found to have non-vanishing viscous stresses in the equilibrium limit, which is counter-intuitive in the physical sense. Because of that, we then discuss the relaxation property of the non-equilibrium stress in the thermodynamic equilibrium limit and its implication on the form of free energy. A modified version of the identical polymer chain model is then proposed, with a special case being the model proposed by G. Holzapfel and J. Simo. Based on the consistent modeling framework, a provably energy stable numerical scheme is constructed for incompressible viscohyperelasticity using inf-sup stable elements. In particular, we adopt a suite of smooth generalization of the Taylor-Hood element based on Non-Uniform Rational B-Splines (NURBS) for spatial discretization. The temporal discretization is performed via the generalized- α scheme. We present a suite of numerical results to corroborate the proposed numerical properties, including the nonlinear stability, robustness under large deformation, and the stress accuracy resolved by the higher-order elements. Additionally, the pathological behavior of the original identical polymer chain model is numerically identified with an unbounded energy decaying. This again underlines the importance of demanding vanishing non-equilibrium stress in the equilibrium limit.

Keywords: Continuum mechanics, Gibbs free energy, Viscoelasticity, Incompressible solids, Isogeometric analysis, Nonlinear stability

1. Introduction

Many polymeric and biological materials may undergo large deformations, with mechanical behavior characterized by hyperelasticity with intrinsic viscous dissipative mechanisms [1, 2, 3, 4, 5]. To incorporate

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the viscous dissipative mechanism into the solid model, there have been several different modeling approaches developed to date. From a big picture perspective, prior modeling work can be categorized into at least two different groups: one based on the concept of internal state variables [6, 7, 8] and one based on the hereditary integral [9, 10, 11, 12]. The former approach invokes a set of internal state variables, which is associated with the physical process occurring at the microscopic level and is manifested at the macroscopic scale. The evolution of the internal state variables is constrained by the second law of thermodynamics and generally results in dissipative behavior. This concept has been successfully applied to modeling inelasticity and phase transitions in general [7, 8]. In the second approach, the model works directly on the stress-strain relationship. The viscous effect is modeled via convolution of a relaxation function with the strain history to describe the fading memory effect on the stress [13], and the convolution is also known as the hereditary integral. This approach includes the notable quasi-linear viscoelasticity theory that has been implemented by the finite element method [14] and applied to investigating soft tissues [15, Chapter 7]. It has also been generalized by utilizing fractional-order derivatives to capture the continuous relaxation spectrum, giving rise to a propitious alternative candidate for viscoelasticity modeling [16, 17, 18]. The constrained mixture theory, which has been developed and applied for vascular growth and remodeling, can be categorized into this general modeling framework, in which the fading memory effect due to the continual turnover is fitted into the hereditary integral [19, 20, 21]. We also mention that some formulations based on internal state variables can be equivalently written into the hereditary integral formulation. The model discussed in this article, as will be shown, can be represented in terms of the hereditary integral with exponential relaxation kernels [22, 23].

Focusing on the internal state variable approach, different underlying modeling assumptions have given rise to different models and computational procedures for viscoelasticity at finite strains. In recent works, it has been common to employ a *multiplicative* decomposition of the deformation gradient to characterize elastic and viscous deformations [24]. With that, viscoelastic materials at finite strains may be established [25, 26, 27, 28, 29, 30, 31], and when combined with Ogden’s hyperelastic function [32], are able to describe responses that may adequately deviate from the thermodynamic equilibrium. Among those works, the model constructed by Reese and Govindjee [26, 27] is rather representative. In their works, the non-equilibrium stresses are consistently derived from a Helmholtz free energy, that can be additively split into equilibrium and non-equilibrium parts [33]. The derivation is similar to that of the finite strain elastoplasticity theory [34], where material isotropy was assumed. Recently, with the isotropy assumption released, the modeling approach of [26] was further extended to account for material anisotropy [35, 36].

Alternatively, a model proposed by Simo in [37] gained popularity over the years and has inspired numerous subsequent modeling works [38, 39, 40, 41, 42]. In his approach, the non-equilibrium stresses due to the viscous effect, which are also termed the “over-stress”, are governed by a set of linear evolution equations, with the elastic stress derived from strain energy. The evolution equations were proposed in a *heuristic* manner as a straightforward generalization of [the standard linear solid model \[43\]](#). [Interested readers are referred to the original work of Maxwell \[44\] and Zener \[45, 46\] for background and further details about the standard linear solid model.](#) Due to the linear nature, the modeling approach is often termed as *finite deformation linear viscoelasticity*, or *finite linear viscoelasticity* for short. The linear evolution equations also enable one to express the non-equilibrium stresses in terms of a simple convolution integral, which can then be computed via a one-step second-order accurate recurrence formula. The algorithm incrementally integrates the constitutive laws by only using the information from the past one step [37, 47, 48]. Thanks to this recurrence formula, the viscoelasticity model becomes amenable to finite element implementations [47, 49, 50]. Another appealing feature is that the strain energy function may conveniently account for material anisotropy [40, 51, 52, 53]. On the other side, although large deformation is allowed for such models, the inherent linearity assumed for the evolution equation is regarded as a drawback of this approach, as it applies only for small deviations from thermodynamic equilibrium [49, Section 6.10] (see also [54, Chapter 10]). Yet, at least under physiological settings, we may reasonably expect the small deviation assumption to remain sound, and thus the model may still be well-suited to many biomechanical applications. A more critical issue is its lack of a rational thermodynamic foundation. Indeed, the linear evolution equations have not been derived from a rational thermomechanical theory, although some related discussions were made in [51]. In recent work, a finite-time blow-up solution has been identified for this

type of model [55], signifying its inconsistent nature. This alerting evidence hinders further adoption of this model for viscoelastic materials.

In this work, we consider a general continuum formulation for fully coupled thermomechanical models with viscous-type dissipation. The viscous deformation is characterized by a set of internal state variables. Unlike the prior approach [37], the non-equilibrium stresses are distinguished from the variables conjugating to the internal state variables. A set of fully nonlinear evolution equations for those conjugate variables are also obtained following the standard Coleman-Noll argument [56]. The origin and relations of the conjugate variables with the non-equilibrium stresses are elucidated through a careful derivation. In particular, it is shown that the non-equilibrium stresses and the conjugate variables are identical only under a very special circumstance, an issue that has long been ignored in the literature [37, 47, 49]. Next, we consider a special form of the non-equilibrium part of the free energy, or the configurational free energy, which governs the viscous responses. The particular form of the energy can be viewed as a generalization of the form proposed in [51]. If one further demands the energy to be quadratic in terms of the internal state variables, a set of linear evolution equations for the conjugate variables is obtained, without invoking any linearization technique [26]. To the best of our knowledge, this is the first time that the thermodynamic origin of these evolution equations gets elucidated. Importantly, the right-hand side of the evolution equations is governed by the *fictitious* second Piola-Kirchhoff stress, which is slightly different from the classical model [47, Chapter 10]. As the first instantiation of this framework, we considered the identical polymer chain model proposed in [51]. Interestingly, it can be easily seen that the non-equilibrium stresses of this model do not relax to zero in the thermodynamic equilibrium limit, which is physically counter-intuitive. It is indeed a requirement that the viscous stresses should vanish for static processes in general [54, Chapter 10]. Without a prior multiplicative decomposition of the deformation, the non-equilibrium stresses are not guaranteed to fully relax in the limit. Therefore, to characterize the relaxation, we discuss the necessary and sufficient conditions from the perspective of the free energy design. Following that, we analyze first a simple model that has been considered in [51, Section 4.2]. This model can be viewed as an extension of the St. Venant-Kirchhoff model to the viscous part and is shown to be the only model in which the non-equilibrium stresses equal the corresponding conjugate variables. Furthermore, the configurational free energy of this model suggests that there is an *additive* split of the quadratic strain, making it parallel to the elastoplasticity theory developed by Green and Naghdi [57, 58]. The last example considered is a modified version of the identical polymer chain model, inspired from the aforesaid one [41, 51] and is guaranteed to have the non-equilibrium stresses fully relaxed in the thermodynamic equilibrium limit.

For finite deformation problems, the deformation is typically highly isochoric. If one considers the Helmholtz free energy as the thermodynamic potential, the resulting system becomes singular in the incompressible limit (see Figure 1). The pressure-volume curve, as part of the stress-strain curve, has a very large slope and becomes a vertical line in the incompressible limit. The value of the slope enters into the elasticity tensor and engenders an ill-conditioned stiffness matrix in the incompressible limit, a well-known issue in the displacement-based elasticity formulation [59]. To handle this issue, one may consider performing a Legendre transformation for the free energy and switch the independent variable from the volume ratio to its conjugate counterpart, the pressure [60, 21]. In doing so, the problem enjoys a saddle-point nature with the pressure acting as an independent variable. Correspondingly, in the incompressible limit, the volume-pressure curve becomes horizontal with a slope approaching zero. With this Legendre transformation performed, the resulting thermodynamic potential becomes the Gibbs free energy, and it is, therefore, a well-suited potential for thermomechanical analysis. This argument is supported by an analysis made in the realm of computational fluid dynamics (CFD). It has been shown that the pressure primitive variable is among the two sets of variables for interpolation if one wants to have a compressible CFD code that survives in the low Mach number limit [61, 62]. The aforementioned Gibbs free energy can be viewed as the thermodynamic explanation of the effectiveness of the pressure primitive variables. A theory for finite elasticity based on the Gibbs free energy was established [60]; the saddle-point nature has also been exploited to design preconditioners [63, 64]; it can also be shown to enjoy a provably nonlinearly stable semi-discrete formulation using inf-sup stable elements [65]; it can be conveniently utilized to construct a strongly-coupled fluid-structure interaction formulation [60, 66]. In this work, we follow our prior approach and use the Gibbs free energy to construct a mixed formulation for viscoelastodynamics. The thermodynamic consistent nature of the

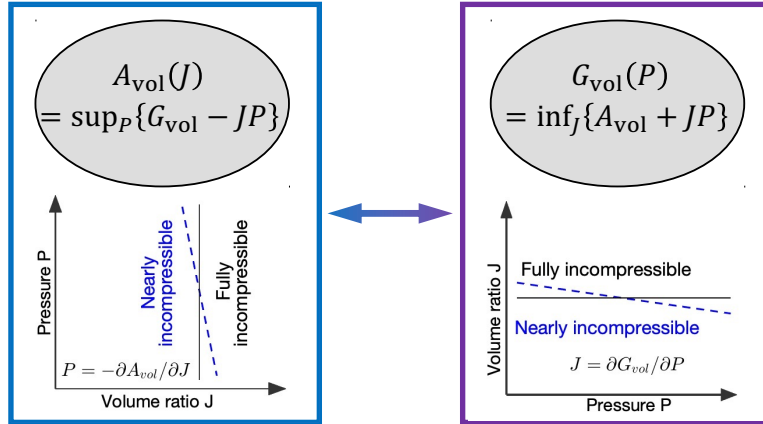


Figure 1: Illustration of the Legendre transformation between the volumetric Helmholtz free energy A_{vol} and the volumetric Gibbs free energy G_{vol} and the resulting constitutive laws based on the two potentials. The Helmholtz free energy $A_{\text{vol}}(J)$ can be transformed to $G_{\text{vol}}(P)$ with P being the thermodynamic pressure, assuming convexity of A_{vol} with respect to the volume ratio J . In the theory based on the Helmholtz free energy, the pressure-volume (i.e., stress-strain) curve becomes a vertical line in the incompressible limit, and the corresponding components in the elasticity tensor blow up to infinity, which in turn causes numerical issues. This partly explains that, for most displacement-based formulations, the tangent matrix is ill-conditioned and demands a direct solver oftentimes. In the constitutive laws based on G_{vol} , the volume ratio is given by $J = \partial G_{\text{vol}} / \partial P$, which becomes a horizontal line in the incompressible limit. The resulting continuum model constitutes a saddle-point problem and requires an inf-sup stable or stabilized discretization technique.

continuum model naturally allows us to devise a discretization that inherits the energy stability property. The resulting mixed formulation necessitates inf-sup stable element pairs, and we adopt a suite of smooth generalization of the Taylor-Hood element by NURBS that has been numerically demonstrated to be stable [65, 67, 68, 69]. In particular, the same set of basis functions is utilized to represent the geometry and to construct the spaces for the displacement and velocity fields, which fits well into the paradigm of NURBS-based isogeometric analysis [70, 71]. One appealing feature of this choice is that it achieves higher-order accuracy [72] without sacrificing robustness [73], in contrast to conventional C^0 -continuous finite elements that often becomes “fragile” when raising the polynomial order [74]. Lastly, we mention other promising candidates for the spatial discretization, such as the smooth generalization of the Raviart-Thomas elements [67, 75] as well as divergence-free discontinuous Galerkin methods [76].

The body of this work starts in Section 2 where we derive a general continuum theory for viscoelasticity with the Gibbs free energy as the thermodynamic potential. Following, we restrict the discussion to a special form of the free energy and provide a definition for the finite linear viscoelasticity. After revisiting the identical polymer chain model, the relaxation condition for the viscous stress is discussed. Two material models are then presented, which completes the continuum modeling section. In Section 3, we consider the spatial and temporal discretization of the constructed continuum model. In particular, we show the energy stability of the proposed numerical formulation. In Section 4, a suite of three numerical tests is performed as an examination of the continuum model and verification of the numerical scheme. We give concluding remarks in Section 5. In Appendices, the implementation details of the three material models are presented.

2. Continuum Basis

2.1. Kinematics

Let $\Omega_{\mathbf{X}}$ and $\Omega_{\mathbf{x}}^t$ be bounded open sets in \mathbb{R}^3 with Lipschitz boundaries. The motion of the body is described by a family of smooth mappings parameterized by the time coordinate t ,

$$\varphi_t(\cdot) = \varphi(\cdot, t) : \Omega_{\mathbf{X}} \rightarrow \Omega_{\mathbf{x}}^t = \varphi(\Omega_{\mathbf{X}}, t) = \varphi_t(\Omega_{\mathbf{X}}), \quad \forall t \geq 0, \quad \mathbf{X} \mapsto \mathbf{x} = \varphi(\mathbf{X}, t) = \varphi_t(\mathbf{X}), \quad \forall \mathbf{X} \in \Omega_{\mathbf{X}}.$$

In the above, \mathbf{x} represents the current position of a material particle originally located at \mathbf{X} , which implies $\varphi(\mathbf{X}, 0) = \mathbf{X}$. The displacement and velocity of the material particle are defined as

$$\mathbf{U} := \varphi(\mathbf{X}, t) - \varphi(\mathbf{X}, 0) = \varphi(\mathbf{X}, t) - \mathbf{X}, \quad \mathbf{V} := \left. \frac{\partial \varphi}{\partial t} \right|_{\mathbf{X}} = \left. \frac{\partial \mathbf{U}}{\partial t} \right|_{\mathbf{X}} = \frac{d\mathbf{U}}{dt}.$$

In this work, we use $d(\cdot)/dt$ to denote a total time derivative. The spatial velocity is defined as $\mathbf{v} := \mathbf{V} \circ \varphi_t^{-1}$. Analogously, we define $\mathbf{u} := \mathbf{U} \circ \varphi_t^{-1}$. The deformation gradient, the Jacobian determinant, and the right Cauchy-Green tensor are defined as

$$\mathbf{F} := \frac{\partial \varphi}{\partial \mathbf{X}}, \quad J := \det(\mathbf{F}), \quad \mathbf{C} := \mathbf{F}^T \mathbf{F}.$$

The Jacobian determinant J is sometimes referred to as the volume ratio as well. Since most materials of interest [behave](#) differently in bulk and shear under large strains, the deformation is multiplicatively decomposed into a volumetric part $J^{1/3} \mathbf{I}$ and an isochoric part $\tilde{\mathbf{F}} := J^{-1/3} \mathbf{F}$ [77, 78]. Clearly, by construction one has the multiplicative decomposition of the deformation gradient as

$$\mathbf{F} = \left(J^{1/3} \mathbf{I} \right) \tilde{\mathbf{F}}. \quad (2.1)$$

The corresponding modified right Cauchy-Green tensor $\tilde{\mathbf{C}}$ is defined as

$$\tilde{\mathbf{C}} := J^{-2/3} \mathbf{C}.$$

To facilitate the following discussion, we note the differentiation relation

$$\frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}} = J^{-2/3} \mathbb{P}^T \quad \text{with} \quad \mathbb{P} := \mathbb{I} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{C},$$

wherein \mathbb{I} is the fourth-order identity tensor. The projection tensor \mathbb{P} furnishes deviatoric behaviors in the Lagrangian description [49]. Furthermore, in this work, the magnitude of a second-order tensor \mathbf{A} is denoted as

$$|\mathbf{A}|^2 := \text{tr}[\mathbf{A} \mathbf{A}^T]. \quad (2.2)$$

2.2. Balance equations and constitutive relations

The motion of the continuum body has to satisfy the local balance equations as well as the second law of thermodynamics [79, 80]. The advective form of the mass balance equation can be written as

$$\frac{d\rho}{dt} + \rho \nabla_{\mathbf{x}} \cdot \mathbf{v} = 0, \quad (2.3)$$

wherein ρ is the mass density in the current configuration. This mass balance equation is equivalent to the equation for the volume ratio J ,

$$\frac{dJ}{dt} = J \nabla_{\mathbf{x}} \cdot \mathbf{v}, \quad \text{or equivalently,} \quad \frac{dJ}{dt} = J \nabla_{\mathbf{X}} \mathbf{V} : \mathbf{F}^{-T}. \quad (2.4)$$

The balance of linear momentum can be written as

$$\rho \frac{d\mathbf{v}}{dt} = \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + \rho \mathbf{b}, \quad \text{or equivalently,} \quad \rho_0 \frac{d\mathbf{V}}{dt} = \nabla_{\mathbf{X}} \cdot \mathbf{P} + \rho_0 \mathbf{B}, \quad (2.5)$$

wherein $\boldsymbol{\sigma}$ denote the Cauchy stress, \mathbf{b} represents the body force per unit mass, $\mathbf{P} := J \boldsymbol{\sigma} \mathbf{F}^{-T}$ is the first Piola-Kirchhoff stress, ρ_0 is the reference mass density, and $\mathbf{B} := \mathbf{b} \circ \varphi_t$. The balance of angular momentum

is satisfied by imposing symmetry on the Cauchy stress, i.e., $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. The balance of internal energy is stated as

$$\rho \frac{d\iota}{dt} = \boldsymbol{\sigma} : \nabla_{\mathbf{x}} \mathbf{v} - \nabla_{\mathbf{x}} \cdot \mathbf{q} + \rho r, \quad (2.6)$$

in which ι is the internal energy per unit mass, \mathbf{q} denote the heat flux, and r is the heat source per unit mass. Further, we introduce the following quantities: the internal energy ι^R defined with respect to the reference volume is related to ι by $\iota^R := \rho_0 \iota$; the heat flux defined per unit reference surface area is denoted as \mathbf{Q} , and it is related to \mathbf{q} by the Piola transformation $\mathbf{Q} = J \mathbf{F}^{-1} \mathbf{q}$; the heat supply per unit reference volume $R := \rho_0 r$. With these definitions, the balance of the internal energy can also be expressed as

$$\frac{d\iota^R}{dt} = \mathbf{P} : \frac{d\mathbf{F}}{dt} - \nabla_{\mathbf{X}} \cdot \mathbf{Q} + R. \quad (2.7)$$

Let us introduce \mathfrak{S} as the entropy per unit volume in the reference configuration, Θ as the absolute temperature field in the reference configuration, and \mathcal{D} as the **total production of entropy**. The second law of thermodynamics states the dissipation of the system is non-negative, i.e.,

$$\mathcal{D} := \frac{d\mathfrak{S}}{dt} + \nabla_{\mathbf{X}} \cdot \left(\frac{\mathbf{Q}}{\Theta} \right) - \frac{R}{\Theta} \geq 0. \quad (2.8)$$

The Gibbs free energy *per unit volume in the reference configuration* is defined as [81, Chapter 5]

$$G := \iota^R - \Theta \mathfrak{S} + PJ, \quad (2.9)$$

wherein P is the thermodynamic pressure defined on the reference configuration, which is the conjugate variable to J . The thermodynamic pressure defined on the current configuration is denoted by $p := P \circ \boldsymbol{\varphi}_t^{-1}$. Taking material time derivatives at both sides of (2.9) results in

$$\frac{dG}{dt} + \mathfrak{S} \frac{d\Theta}{dt} - J \frac{dP}{dt} = \frac{d\iota^R}{dt} - \Theta \frac{d\mathfrak{S}}{dt} + P \frac{dJ}{dt}.$$

Substituting the balance equation of the internal energy (2.7), the second law of thermodynamics (2.8), and the mass balance equation (2.4) into the above equation, one readily obtains

$$\Theta \mathcal{D} = J \boldsymbol{\sigma} : \nabla_{\mathbf{x}} \mathbf{v} - \frac{\mathbf{Q} \cdot \nabla_{\mathbf{X}} \Theta}{\Theta} + PJ \nabla_{\mathbf{x}} \cdot \mathbf{v} - \mathfrak{S} \frac{d\Theta}{dt} + J \frac{dP}{dt} - \frac{dG}{dt}. \quad (2.10)$$

We may additively split the Cauchy stress into deviatoric and hydrostatic parts,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{dev}} + \frac{1}{3} (\text{tr} [\boldsymbol{\sigma}]) \mathbf{I},$$

with which one may show that

$$J \boldsymbol{\sigma} : \nabla_{\mathbf{x}} \mathbf{v} = \frac{1}{2} J \tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T} : \frac{d\tilde{\mathbf{C}}}{dt} + \frac{1}{3} J \text{tr} [\boldsymbol{\sigma}] \nabla_{\mathbf{x}} \cdot \mathbf{v}.$$

Consequently, the relation (2.10) can be rewritten as

$$\Theta \mathcal{D} = \frac{1}{2} J \tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T} : \frac{d\tilde{\mathbf{C}}}{dt} + J \left(\frac{1}{3} \text{tr} [\boldsymbol{\sigma}] + p \right) \nabla_{\mathbf{x}} \cdot \mathbf{v} - \frac{\mathbf{Q} \cdot \nabla_{\mathbf{X}} \Theta}{\Theta} - \mathfrak{S} \frac{d\Theta}{dt} + J \frac{dP}{dt} - \frac{dG}{dt}. \quad (2.11)$$

From the above, we postulate that the Gibbs free energy G is a function of $\tilde{\mathbf{C}}$, P , and Θ by invoking Truesdell's principle of equipresence [82]. We assume that G additionally depends on a set of strain-like internal state variables $\{\boldsymbol{\Gamma}^\alpha\}_{\alpha=1}^m$,

$$G = G(\tilde{\mathbf{C}}, P, \Theta, \boldsymbol{\Gamma}^1, \dots, \boldsymbol{\Gamma}^m). \quad (2.12)$$

Here m is the number of relaxation processes characterizing the viscous response of the material. Oftentimes, the internal state variables $\mathbf{\Gamma}^\alpha$ is viewed as a deformation tensor akin to the right Cauchy-Green tensor [38, 51]. We demand that for a given homogeneous reference temperature $\Theta_0 > 0$,

$$G(\mathbf{I}, \theta, \Theta_0, \mathbf{I}, \dots, \mathbf{I}) = 0, \quad (2.13)$$

which is commonly known as the normalization condition [49]. With the above function form of the Gibbs free energy, material time derivative of the Gibbs free energy can be written explicitly as

$$\frac{dG}{dt} = \frac{1}{2} \tilde{\mathbf{S}} : \frac{d\tilde{\mathbf{C}}}{dt} + \frac{\partial G}{\partial P} \frac{dP}{dt} + \frac{\partial G}{\partial \Theta} \frac{d\Theta}{dt} + \sum_{\alpha=1}^m \frac{\partial G}{\partial \mathbf{\Gamma}^\alpha} : \frac{d\mathbf{\Gamma}^\alpha}{dt}, \quad (2.14)$$

in which we introduced the fictitious second Piola-Kirchhoff stress

$$\tilde{\mathbf{S}} := 2 \frac{\partial G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial \tilde{\mathbf{C}}}.$$

Substituting (2.14) into (2.10) leads to

$$\begin{aligned} \Theta \mathcal{D} = & \left(\frac{1}{2} J \tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T} - \frac{1}{2} \tilde{\mathbf{S}} \right) : \frac{d\tilde{\mathbf{C}}}{dt} + J \left(\frac{1}{3} \text{tr}[\boldsymbol{\sigma}] + p \right) \nabla_{\mathbf{x}} \cdot \mathbf{v} - \frac{\mathcal{Q} \cdot \nabla_{\mathbf{x}} \Theta}{\Theta} \\ & - \left(\mathfrak{S} + \frac{\partial G}{\partial \Theta} \right) \frac{d\Theta}{dt} + \left(J - \frac{\partial G}{\partial P} \right) \frac{dP}{dt} - 2 \sum_{\alpha=1}^m \frac{\partial G}{\partial \mathbf{\Gamma}^\alpha} : \frac{1}{2} \frac{d\mathbf{\Gamma}^\alpha}{dt}. \end{aligned} \quad (2.15)$$

We introduce \mathbf{Q}^α as the conjugate variables to the internal state variables $\mathbf{\Gamma}^\alpha$,

$$\mathbf{Q}^\alpha := -2 \frac{\partial G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial \mathbf{\Gamma}^\alpha}. \quad (2.16)$$

Noticing that

$$\frac{d\tilde{\mathbf{C}}}{dt} = \frac{\partial \tilde{\mathbf{C}}}{\partial \mathbf{C}} : \frac{d\mathbf{C}}{dt} = J^{-\frac{2}{3}} \mathbb{P}^T : \frac{d\mathbf{C}}{dt},$$

we may reorganize (2.15) as follows,

$$\begin{aligned} \Theta \mathcal{D} = & \left(\frac{1}{2} J \tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T} - \frac{1}{2} \tilde{\mathbf{S}} \right) : \left(J^{-\frac{2}{3}} \mathbb{P}^T : \frac{d\mathbf{C}}{dt} \right) + J \left(\frac{1}{3} \text{tr}[\boldsymbol{\sigma}] + p \right) \nabla_{\mathbf{x}} \cdot \mathbf{v} - \frac{\mathcal{Q} \cdot \nabla_{\mathbf{x}} \Theta}{\Theta} \\ & - \left(\mathfrak{S} + \frac{\partial G}{\partial \Theta} \right) \frac{d\Theta}{dt} + \left(J - \frac{\partial G}{\partial P} \right) \frac{dP}{dt} + \sum_{\alpha=1}^m \mathbf{Q}^\alpha : \frac{1}{2} \frac{d\mathbf{\Gamma}^\alpha}{dt} \\ = & J^{-\frac{2}{3}} \left(\frac{1}{2} J \tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T} - \frac{1}{2} \mathbb{P} : \tilde{\mathbf{S}} \right) : \frac{d\mathbf{C}}{dt} + J \left(\frac{1}{3} \text{tr}[\boldsymbol{\sigma}] + p \right) \nabla_{\mathbf{x}} \cdot \mathbf{v} - \frac{\mathcal{Q} \cdot \nabla_{\mathbf{x}} \Theta}{\Theta} \\ & - \left(\mathfrak{S} + \frac{\partial G}{\partial \Theta} \right) \frac{d\Theta}{dt} + \left(J - \frac{\partial G}{\partial P} \right) \frac{dP}{dt} + \sum_{\alpha=1}^m \mathbf{Q}^\alpha : \frac{1}{2} \frac{d\mathbf{\Gamma}^\alpha}{dt} \end{aligned} \quad (2.17)$$

The second equality of the above derivation makes use of the fact that

$$\mathbb{P} : \left(\tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T} \right) = \tilde{\mathbf{F}}^{-1} \boldsymbol{\sigma}_{\text{dev}} \tilde{\mathbf{F}}^{-T}.$$

The relation (2.17) furnishes a natural means of choosing constitutive relations that ensure the satisfaction of the second law of thermodynamics. We may thereby make the following choices for the constitutive relations,

$$\boldsymbol{\sigma}_{\text{dev}} = J^{-1} \tilde{\mathbf{F}} \left(\mathbb{P} : \tilde{\mathbf{S}} \right) \tilde{\mathbf{F}}^T = J^{-1} \tilde{\mathbf{F}} \left(\mathbb{P} : \left(2 \frac{\partial G}{\partial \tilde{\mathbf{C}}} \right) \right) \tilde{\mathbf{F}}^T, \quad (2.18)$$

$$\frac{1}{3} \operatorname{tr} [\boldsymbol{\sigma}] = -p, \quad (2.19)$$

$$\boldsymbol{Q} = -\bar{\kappa} \nabla_{\mathbf{X}} \Theta, \quad (2.20)$$

$$\mathfrak{S} = -\frac{\partial G}{\partial \Theta}, \quad (2.21)$$

$$\rho = \rho_0 \left(\frac{\partial G}{\partial P} \right)^{-1}, \quad (2.22)$$

$$\mathbf{Q}^\alpha = \mathbb{V}^\alpha : \left(\frac{1}{2} \frac{d}{dt} \boldsymbol{\Gamma}^\alpha \right). \quad (2.23)$$

In the above, $\bar{\kappa}$ is the thermal conductivity and \mathbb{V}^α is a positive definite fourth-order viscosity tensor. Here, we also assume that there exists a fourth-order tensor $(\mathbb{V}^\alpha)^{-1}$ such that an inverse relation for (2.23) holds,

$$\frac{1}{2} \frac{d}{dt} \boldsymbol{\Gamma}^\alpha = (\mathbb{V}^\alpha)^{-1} : \mathbf{Q}^\alpha. \quad (2.24)$$

Based on (2.18) and (2.19), the Cauchy stress $\boldsymbol{\sigma}$ can be represented as

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}_{\text{dev}} + \frac{1}{3} \operatorname{tr} [\boldsymbol{\sigma}] \mathbf{I} = J^{-1} \tilde{\mathbf{F}} \left(\mathbb{P} : \tilde{\mathbf{S}} \right) \tilde{\mathbf{F}}^T - p \mathbf{I} = J^{-1} \tilde{\mathbf{F}} \left(\mathbb{P} : \left(2 \frac{\partial G}{\partial \tilde{\mathbf{C}}} \right) \right) \tilde{\mathbf{F}}^T - p \mathbf{I}.$$

Accordingly, the second Piola-Kirchhoff stress \mathbf{S} , as the pull-back operation performed on $J\boldsymbol{\sigma}$, can be written as

$$\begin{aligned} \mathbf{S} &:= J \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}^{-T} = J \mathbf{F}^{-1} \left(\boldsymbol{\sigma}_{\text{dev}} + \frac{1}{3} \operatorname{tr} [\boldsymbol{\sigma}] \mathbf{I} \right) \mathbf{F}^{-T} = \mathbf{S}_{\text{iso}} + \mathbf{S}_{\text{vol}}, \\ \mathbf{S}_{\text{iso}} &:= J \mathbf{F}^{-1} \boldsymbol{\sigma}_{\text{dev}} \mathbf{F}^{-T} = J^{-\frac{2}{3}} \mathbb{P} : \tilde{\mathbf{S}} = J^{-\frac{2}{3}} \mathbb{P} : 2 \frac{\partial G}{\partial \tilde{\mathbf{C}}}, \\ \mathbf{S}_{\text{vol}} &:= \frac{1}{3} \operatorname{tr} [\boldsymbol{\sigma}] J \mathbf{F}^{-1} \mathbf{F}^{-T} = -J P C^{-1}. \end{aligned}$$

With the above choices, it can be shown that the dissipation relation (2.15) reduces to

$$\begin{aligned} \Theta \mathcal{D} &= \bar{\kappa} |\nabla_{\mathbf{X}} \Theta|^2 - \sum_{\alpha=1}^m \mathbf{Q}^\alpha : \frac{1}{2} \left(\frac{d}{dt} \boldsymbol{\Gamma}^\alpha \right) = \bar{\kappa} |\nabla_{\mathbf{X}} \Theta|^2 + \sum_{\alpha=1}^m \frac{1}{4} \left(\frac{d}{dt} \boldsymbol{\Gamma}^\alpha \right) : \mathbb{V}^\alpha : \left(\frac{d}{dt} \boldsymbol{\Gamma}^\alpha \right) \\ &= \bar{\kappa} |\nabla_{\mathbf{X}} \Theta|^2 + \sum_{\alpha=1}^m \mathbf{Q}^\alpha : (\mathbb{V}^\alpha)^{-1} : \mathbf{Q}^\alpha, \end{aligned} \quad (2.25)$$

which remains non-negative for arbitrary processes, thereby automatically satisfying the second law of thermodynamics. The choices of the constitutive relations (2.18)-(2.23) are certainly not a unique way for ensuring the second law of thermodynamics. Yet, as will be shown, these choices are general enough to describe the mechanical behavior of a variety of materials. The definition of \mathbf{Q}^α (2.16) suggests that it is a function of $\boldsymbol{\Gamma}^\alpha$, $\tilde{\mathbf{C}}$, P , and Θ . Thereby, taking material time derivatives of \mathbf{Q}^α yields

$$\frac{d}{dt} \mathbf{Q}^\alpha = 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \boldsymbol{\Gamma}^\alpha} : \frac{1}{2} \frac{d}{dt} \boldsymbol{\Gamma}^\alpha + 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \tilde{\mathbf{C}}} : \frac{1}{2} \frac{d}{dt} \tilde{\mathbf{C}} + \frac{\partial \mathbf{Q}^\alpha}{\partial \Theta} \frac{d}{dt} \Theta + \frac{\partial \mathbf{Q}^\alpha}{\partial P} \frac{d}{dt} P.$$

Invoking the constitutive relation (2.24) to replace the rate of $\boldsymbol{\Gamma}^\alpha$ in the above relation, we obtain a set of evolution equations for \mathbf{Q}^α after rearranging terms,

$$\frac{d}{dt} \mathbf{Q}^\alpha - 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \boldsymbol{\Gamma}^\alpha} : (\mathbb{V}^\alpha)^{-1} : \mathbf{Q}^\alpha = 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \tilde{\mathbf{C}}} : \frac{1}{2} \frac{d}{dt} \tilde{\mathbf{C}} + \frac{\partial \mathbf{Q}^\alpha}{\partial \Theta} \frac{d}{dt} \Theta + \frac{\partial \mathbf{Q}^\alpha}{\partial P} \frac{d}{dt} P. \quad (2.26)$$

The above constitutes a set of general evolution equations for \mathbf{Q}^α , and in the latter part of this work we will reveal how it may recover the familiar linear evolution equations [49, Chapter 6]. In the last, we characterize the thermodynamic equilibrium state by the following two conditions,

$$\nabla_{\mathbf{X}}\Theta\Big|_{\text{eq}} = \mathbf{0}, \quad \text{and} \quad \frac{d}{dt}\mathbf{\Gamma}^\alpha\Big|_{\text{eq}} = \mathbf{0}, \quad \text{for } \alpha = 1, \dots, m. \quad (2.27)$$

In the above, the notation $\Big|_{\text{eq}}$ indicates that equality holds at the thermodynamic equilibrium limit. The conditions (2.27) suggest that, in the equilibrium limit, there is no more dissipation in the system. In particular, (2.27)₂ implies that there is no more change of the internal state variables in the thermodynamic equilibrium limit. According to the constitutive relation (2.23), (2.27)₂ is equivalent to

$$\mathbf{Q}^\alpha\Big|_{\text{eq}} = \mathbf{0}, \quad \text{for } \alpha = 1, \dots, m. \quad (2.28)$$

2.3. An additive split of the free energy and refined constitutive relations

Here we consider a more refined structure of the thermodynamic potential G . First, noting that the modified right Cauchy-Green tensor describes volume-preserving deformations, one may reasonably demand that the density is a function of the pressure and temperature. Then the constitutive relation (2.22) implies that

$$\frac{\partial G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial P} = \rho_0 \rho^{-1}(p, \theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m), \quad (2.29)$$

with the equality holds by recalling that $P(\mathbf{X}) = p(\boldsymbol{\varphi}_t(\mathbf{X}), t)$ and $\Theta(\mathbf{X}) = \theta(\boldsymbol{\varphi}_t(\mathbf{X}), t)$. The relation (2.29) suggests that the free energy can be additively split as

$$G = G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = G_{\text{vol}}^\infty(P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) + G_{\text{iso}}(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m), \quad (2.30)$$

by performing a partial integration of (2.29) with respect to P (see also [60, p. 559]). This split structure of the free energy has also been justified based on physical observations [83, 84]. It needs to be pointed out that the split is a logical consequence of the multiplicative decomposition (2.1). **Second, experimental data shows that the bulk viscoelastic behavior is typically several orders of magnitude narrower than the shear viscoelastic behavior [1, Chapter 18]. Therefore, it is reasonable to model viscous effects only in the isochoric motion, and we consider the volumetric energy G_{vol}^∞ to be independent of the internal state variables.** Third, as a commonly adopted approach [49], the isochoric part of the free energy G_{iso} can be further split into an equilibrium part as well as a dissipative, or non-equilibrium, part. The last two assumptions further lead to the following form of the Gibbs free energy,

$$G = G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = G_{\text{vol}}^\infty(P, \Theta) + G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta) + \sum_{\alpha=1}^m \Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m), \quad (2.31)$$

The two terms G_{vol}^∞ and G_{iso}^∞ characterize the hyperelastic material behavior at the equilibrium state as time approaches infinity. Given the explicit structure of the Gibbs free energy (2.31), we may introduce the equilibrium and non-equilibrium parts of the fictitious second Piola-Kirchhoff stress as

$$\tilde{\mathbf{S}} = \tilde{\mathbf{S}}_{\text{iso}}^\infty + \sum_{\alpha=1}^m \tilde{\mathbf{S}}_{\text{neq}}^\alpha, \quad \tilde{\mathbf{S}}_{\text{iso}}^\infty := 2 \frac{\partial G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}}, \quad \tilde{\mathbf{S}}_{\text{neq}}^\alpha := 2 \frac{\partial \Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial \tilde{\mathbf{C}}}. \quad (2.32)$$

Correspondingly, the isochoric part of the second Piola-Kirchhoff stress can be expressed as

$$\mathbf{S}_{\text{iso}} = \mathbf{S}_{\text{iso}}^\infty + \sum_{\alpha=1}^m \mathbf{S}_{\text{neq}}^\alpha, \quad \mathbf{S}_{\text{iso}}^\infty := J^{-\frac{2}{3}} \mathbb{P} : \tilde{\mathbf{S}}_{\text{iso}}^\infty, \quad \mathbf{S}_{\text{neq}}^\alpha := J^{-\frac{2}{3}} \mathbb{P} : \tilde{\mathbf{S}}_{\text{neq}}^\alpha.$$

Due to (2.31), \mathbf{Q}^α can be defined in terms of the configurational free energy Υ^α as

$$\mathbf{Q}^\alpha := -2 \frac{\partial G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial \mathbf{\Gamma}^\alpha} = -2 \sum_{\beta=1}^m \frac{\partial \Upsilon^\beta(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial \mathbf{\Gamma}^\alpha}. \quad (2.33)$$

Correspondingly, the evolution equations (2.26) can be reduced to

$$\frac{d}{dt} \mathbf{Q}^\alpha - 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \mathbf{\Gamma}^\alpha} : (\mathbb{V}^\alpha)^{-1} : \mathbf{Q}^\alpha = 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \tilde{\mathbf{C}}} : \frac{1}{2} \frac{d}{dt} \tilde{\mathbf{C}} + \frac{\partial \mathbf{Q}^\alpha}{\partial \Theta} \frac{d}{dt} \Theta, \quad (2.34)$$

since \mathbf{Q}^α is independent of P from (2.33). Here the evolution equations (2.34) together with (2.33) can be viewed as a set of nonlinear rate equations characterizing the evolution of the internal state variables $\mathbf{\Gamma}^\alpha$ as well. Finally, to complete the thermomechanical theory, the entropy per unit reference volume adopts the form,

$$\mathfrak{S} = \mathfrak{S}^\infty + \sum_{\alpha=1}^m \mathfrak{S}^\alpha, \quad \mathfrak{S}^\infty := -\frac{\partial G_{\text{vol}}^\infty}{\partial \Theta} - \frac{\partial G_{\text{iso}}^\infty}{\partial \Theta}, \quad \mathfrak{S}^\alpha := -\frac{\partial \Upsilon^\alpha}{\partial \Theta}.$$

Remark 1. In our derivation, we start from the classical definition of the Gibbs free energy (2.9) without assuming which variables it should depend on. Subsequently, the derived relation (2.11) together with the Truesdell's principle of equipresence suggests that it should be a function of $\tilde{\mathbf{C}}$, P , and Θ . With the additive split of the free energy into isochoric and volumetric parts, we may observe from (2.32)₂ that the isochoric part of the Gibbs free energy plays the same role as the isochoric part of the conventional strain energy of the Helmholtz type [49, Chapter 6]. Therefore, we may view that the Gibbs free energy is related to the Helmholtz free energy by performing the Legendre transformation on the volumetric part of the free energy only [60, Section 2.4]. Indeed, the volumetric energy is commonly convex with respect to J , unless there are multiple stable phases (e.g., the van der Waals model [85, 86]), and this convexity property guarantees the validity of the Legendre transformation. We note that some adopt a different notion of the Gibbs free energy which depends on the whole stress [87, 88]. That type of theory requires the convexity of the free energy with respect to the whole strain, which does not hold for many nonlinear materials [89].

Remark 2. There is a requirement for the non-equilibrium stresses based on physical intuition and consistent with other theories (e.g., the small strain theory and the multiplicative viscoelasticity theory), that is $\mathbf{S}_{\text{neq}}^\alpha = J^{-\frac{2}{3}} \mathbb{P} : \tilde{\mathbf{S}}_{\text{neq}}^\alpha$ should fully relax, or vanish, in the thermodynamic equilibrium limit. This condition apparently poses an additional constraint for the configurational free energy Υ^α , and we will revisit this point in Section 2.4.1.

Remark 3. It has recently been observed that the additive split of the energy (2.30) may lead to non-physical responses for anisotropic materials [84, 90, 91]. In this work, we restrict our discussion to isotropic materials, and the split (2.30) is sound in both mathematics and physics [83, 84].

Remark 4. If we assume the viscosity tensors \mathbb{V}^α are isotropic, they adopt explicit forms as

$$\mathbb{V}^\alpha = 2\eta_D^\alpha(\Theta) \left(\mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) + \frac{2}{3} \eta_V^\alpha(\Theta) \mathbf{I} \otimes \mathbf{I}, \quad (2.35)$$

where $\eta_D^\alpha(\Theta)$ and $\eta_V^\alpha(\Theta)$ are non-negative and represent the deviatoric and volumetric viscosities respectively [26, 92, 93]. The conjugate variables \mathbf{Q}^α are related to the internal state variables $\mathbf{\Gamma}^\alpha$ by a linear relation,

$$\mathbf{Q}^\alpha = \eta_D^\alpha(\Theta) \left(\frac{d}{dt} \mathbf{\Gamma}^\alpha - \frac{1}{3} \text{tr} \left[\frac{d}{dt} \mathbf{\Gamma}^\alpha \right] \mathbf{I} \right) + \frac{1}{3} \eta_V^\alpha(\Theta) \text{tr} \left[\frac{d}{dt} \mathbf{\Gamma}^\alpha \right] \mathbf{I}. \quad (2.36)$$

Consequently, the dissipation relation (2.25) reduces to

$$\Theta \mathcal{D} = \bar{\kappa} |\nabla_{\mathbf{X}} \Theta|^2 + \sum_{\alpha=1}^m \left(\frac{\eta_D^\alpha(\Theta)}{2} \left| \frac{d}{dt} \mathbf{\Gamma}^\alpha - \frac{1}{3} \text{tr} \left[\frac{d}{dt} \mathbf{\Gamma}^\alpha \right] \mathbf{I} \right|^2 + \frac{\eta_V^\alpha(\Theta)}{6} \text{tr} \left[\frac{d}{dt} \mathbf{\Gamma}^\alpha \right]^2 \right).$$

2.4. Finite linear viscoelasticity

In this section, we consider specifically the finite deformation linear viscoelasticity, the precise definition of which will be given momentarily. We start by restricting the configurational free energy to the following form,

$$\Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = H^\alpha(\mathbf{\Gamma}^\alpha, \Theta) + \left(\hat{\mathbf{S}}_0^\alpha - 2 \frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} \right) : \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} + F^\alpha(\tilde{\mathbf{C}}, \Theta). \quad (2.37)$$

In the above choice, the α -th configurational free energy Υ^α does not depend on $\mathbf{\Gamma}^\beta$ for $\beta \neq \alpha$; H^α , G^α , and F^α are scalar-valued functions to be provided for material modeling; $\hat{\mathbf{S}}_0^\alpha$ is a constant stress-like tensor whose physical significance will be revealed in Section 2.4.1. A similar form of the configurational free energy was proposed in [51] with minor differences. To satisfy the normalization condition for Υ^α , we demand that

$$H^\alpha(\mathbf{I}, \Theta_0) = 0, \quad F^\alpha(\mathbf{I}, \Theta_0) = 0,$$

for a given homogeneous temperature $\Theta_0 > 0$. It is worth pointing out that the essence of the particular configurational free energy is that the **modified right Cauchy-Green tensor** $\tilde{\mathbf{C}}$ and the internal state variable $\mathbf{\Gamma}^\alpha$ are coupled through the second term in (2.37) only, where the internal state variables $\mathbf{\Gamma}^\alpha$ are present in linear forms. Also, recalling that $\mathbf{\Gamma}^\alpha$ are akin to the right Cauchy-Green tensor, the terms $(\mathbf{\Gamma}^\alpha - \mathbf{I})/2$ are analogous to the Green-Lagrange strain tensor. It will facilitate our subsequent discussion by introducing a stress term defined as

$$\tilde{\mathbf{S}}_{\text{iso}}^\alpha(\tilde{\mathbf{C}}, \Theta) := 2 \frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}},$$

with which the configurational free energy (2.37) can be rewritten as

$$\Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = H^\alpha(\mathbf{\Gamma}^\alpha, \Theta) + \left(\hat{\mathbf{S}}_0^\alpha - \tilde{\mathbf{S}}_{\text{iso}}^\alpha(\tilde{\mathbf{C}}, \Theta) \right) : \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} + F^\alpha(\tilde{\mathbf{C}}, \Theta).$$

Due to the definition (2.33) and the form of the configurational free energy (2.37), we now have

$$\begin{aligned} \mathbf{Q}^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) &= -2 \frac{\partial \Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m)}{\partial \mathbf{\Gamma}^\alpha} = 2 \left(\frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} - \frac{\partial H^\alpha(\mathbf{\Gamma}^\alpha, \Theta)}{\partial \mathbf{\Gamma}^\alpha} \right) - \hat{\mathbf{S}}_0^\alpha \\ &= \tilde{\mathbf{S}}_{\text{iso}}^\alpha(\tilde{\mathbf{C}}, \Theta) - \hat{\mathbf{S}}_0^\alpha - 2 \frac{\partial H^\alpha(\mathbf{\Gamma}^\alpha, \Theta)}{\partial \mathbf{\Gamma}^\alpha}. \end{aligned} \quad (2.38)$$

From the above relation, the material time derivative of $\tilde{\mathbf{S}}_{\text{iso}}^\alpha$ can be expressed as

$$\frac{d}{dt} \tilde{\mathbf{S}}_{\text{iso}}^\alpha = \frac{\partial \mathbf{Q}^\alpha}{\partial \tilde{\mathbf{C}}} : \frac{d}{dt} \tilde{\mathbf{C}} + \frac{\partial \mathbf{Q}^\alpha}{\partial \Theta} \frac{d}{dt} \Theta + 2 \frac{\partial^2 H^\alpha}{\partial \mathbf{\Gamma}^\alpha \partial \Theta} \frac{d}{dt} \Theta,$$

with which the right-hand side of the evolution equations (2.34) can be written compactly as

$$\frac{d}{dt} \mathbf{Q}^\alpha - 2 \frac{\partial \mathbf{Q}^\alpha}{\partial \mathbf{\Gamma}^\alpha} : (\mathbb{V}^\alpha)^{-1} : \mathbf{Q}^\alpha = \frac{d}{dt} \tilde{\mathbf{S}}_{\text{iso}}^\alpha - 2 \frac{\partial^2 H^\alpha}{\partial \mathbf{\Gamma}^\alpha \partial \Theta} \frac{d}{dt} \Theta. \quad (2.39)$$

The non-equilibrium stresses (2.32)₃ now can be represented as

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = 2 \frac{\partial \Upsilon^\alpha}{\partial \tilde{\mathbf{C}}} = 2 \left(\frac{\partial F^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} - \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right), \quad (2.40)$$

in which we have made use of the major symmetry of the fourth-order tensor $\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha / \partial \tilde{\mathbf{C}}$.

2.4.1. Definition

Based on the above discussion, we now provide the definition of finite linear viscoelasticity here.

Definition 1. A finite linear viscoelastic material is described by the configurational free energy (2.37) with $\mathbb{V}^\alpha = 2\eta^\alpha(\Theta)\mathbb{I}$ and a quadratic form for H^α in terms of the internal strain variables,

$$H^\alpha(\mathbf{\Gamma}^\alpha, \Theta) = \mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2, \quad (2.41)$$

where $\mu^\alpha(\Theta)$ is a temperature dependent shear modulus associated with the α -th relaxation process.

The form $\mathbb{V}^\alpha = 2\eta^\alpha(\Theta)\mathbb{I}$ is a special case for an isotropic tensor of order four by taking $\eta^\alpha = \eta_D^\alpha = \eta_V^\alpha$ in (2.35). According to (2.38), one can show that the quadratic form (2.41) implies

$$\mathbf{Q}^\alpha = \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha - 2\mu^\alpha(\Theta) \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2}, \quad \text{and} \quad \frac{\partial \mathbf{Q}^\alpha}{\partial \mathbf{\Gamma}^\alpha} = -\mu^\alpha(\Theta)\mathbb{I}. \quad (2.42)$$

Apparently, the choice of the quadratic form for H^α leads to the above linear relation between \mathbf{Q}^α and $\mathbf{\Gamma}^\alpha$, which further simplifies the evolution equations (2.39) as

$$\frac{d}{dt} \mathbf{Q}^\alpha + \frac{\mathbf{Q}^\alpha}{\tau^\alpha} = \frac{d}{dt} \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \frac{d\mu^\alpha}{dt} (\mathbf{\Gamma}^\alpha - \mathbf{I}), \quad (2.43)$$

with

$$\tau^\alpha(\Theta) := \eta^\alpha(\Theta) / \mu^\alpha(\Theta).$$

The term τ^α has the dimension of time and is commonly referred to as the relaxation time for the α -th process. For most polymers, the relaxation time τ^α can be modeled by the Arrhenius equation, which characterizes faster viscoelastic effects with the increase of temperature [94]. The second term on the right-hand side of (2.43) arises due to the temperature dependent material parameters [49, p. 365]. Given an initial condition $\mathbf{Q}_0^\alpha := \mathbf{Q}^\alpha|_{t=0}$, the solution of (2.43) can be obtained in a hereditary integral form,

$$\mathbf{Q}^\alpha = \exp(-t/\tau^\alpha) \mathbf{Q}_0^\alpha + \int_{0^+}^t \exp(-(t-s)/\tau^\alpha) \left(\frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^\alpha - (\mathbf{\Gamma}^\alpha - \mathbf{I}) \frac{d\mu^\alpha}{ds} \right) ds. \quad (2.44)$$

We recall that the general evolution equations (2.39) is nonlinear for \mathbf{Q}^α . For finite linear viscoelastic materials, the evolution equations reduce to a linear system, whose solution can be conveniently represented in terms of the hereditary integral form (2.44). Within this work, we assume that $\mathbf{\Gamma}^\alpha = \mathbf{I}$ for $\alpha = 1, \dots, m$ at time $t = 0$. Consequently, the relation (2.42)₁ implies

$$\mathbf{Q}_0^\alpha = \tilde{\mathbf{S}}_{\text{iso } 0}^\alpha - \hat{\mathbf{S}}_0^\alpha, \quad \text{with} \quad \tilde{\mathbf{S}}_{\text{iso } 0}^\alpha := \tilde{\mathbf{S}}_{\text{iso}}^\alpha|_{t=0}.$$

Therefore, the constant tensor $\hat{\mathbf{S}}_0^\alpha$ is determined by the initial values of the stresses,

$$\hat{\mathbf{S}}_0^\alpha := \tilde{\mathbf{S}}_{\text{iso } 0}^\alpha - \mathbf{Q}_0^\alpha, \quad (2.45)$$

which ensures the consistency of the initial condition. With (2.40) and (2.42), the non-equilibrium stresses for finite linear viscoelasticity are given by

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = 2 \left(\frac{\partial F_\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} - \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right) = 2 \frac{\partial F_\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} - \frac{1}{\mu^\alpha(\Theta)} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \left(\tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha - \mathbf{Q}^\alpha \right).$$

Remark 5. The evolution equations (2.43) derived here are similar to those introduced and used in [37, 38, 39, 47, 51, 95], which can be expressed as

$$\frac{d}{dt} \mathbf{Q}^\alpha + \frac{\mathbf{Q}^\alpha}{\tau^\alpha} = \frac{d}{dt} \left(J^{-\frac{2}{3}} \mathbb{P} : \tilde{\mathbf{S}}_{\text{iso}}^\alpha \right) - 2 \frac{d\mu^\alpha}{dt} \mathbf{\Gamma}^\alpha.$$

We emphasize that the evolution equations in those works were proposed based on a purely heuristic argument with inspiration coming from the standard linear solid model [96]. The right-hand side terms of the above evolution equations are different from those in (2.43), in which a deviatoric projected stress from $\tilde{\mathbf{S}}_{\text{iso}}^\alpha$ was utilized to drive the evolution of \mathbf{Q}^α .

Remark 6. The evolution equations (2.43) can be simply derived by taking time derivatives at both sides of (2.42)₁ and invoking the constitutive relations (2.24), and they characterize the evolution of the stress-like variables \mathbf{Q}^α . One may also obtain a set of evolution equations for the strain-like quantities $\mathbf{\Gamma}^\alpha - \mathbf{I}$ by invoking (2.24) directly to replace \mathbf{Q}^α in (2.42)₁, which leads to

$$\eta^\alpha(\Theta) \frac{d}{dt} (\mathbf{\Gamma}^\alpha - \mathbf{I}) + \mu^\alpha(\Theta) (\mathbf{\Gamma}^\alpha - \mathbf{I}) = \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha.$$

This set of evolution equations can be utilized to design an algorithm that tracks the changes of the strain-like internal state variables, leading to a completely strain-driven formulation.

2.4.2. A model based on the identical polymer chain assumption

Now for a finite linear viscoelasticity model, considering the form for H^α given in Definition 1, the modeling work reduces to designing the form of G^α and F^α in (2.37). In [41, 51], a model was proposed by taking

$$G^\alpha(\tilde{\mathbf{C}}, \Theta) = F^\alpha(\tilde{\mathbf{C}}, \Theta) = \beta_\infty^\alpha G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta),$$

with $\beta_\infty^\alpha \in (0, \infty)$ being non-dimensional constants. This model is motivated by the observation that the phenomenological viscoelastic behavior is induced by a medium composed of identical polymer chains. With this assumption, the viscoelastic material is completely characterized by the potential G_{iso}^∞ together with non-dimensional parameters β_∞^α . The configurational free energy for this model can be rewritten as

$$\mathcal{R}^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = \mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2 + \left(\hat{\mathbf{S}}_0^\alpha - \beta_\infty^\alpha \tilde{\mathbf{S}}_{\text{iso}}^\infty \right) : \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} + \beta_\infty^\alpha G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta). \quad (2.46)$$

The evolution equations can be expressed as

$$\frac{d}{dt} \mathbf{Q}^\alpha + \frac{\mathbf{Q}^\alpha}{\tau^\alpha} = \beta_\infty^\alpha \frac{d}{dt} \tilde{\mathbf{S}}_{\text{iso}}^\infty - \frac{d\mu^\alpha}{dt} (\mathbf{\Gamma}^\alpha - \mathbf{I}). \quad (2.47)$$

With \mathbf{Q}^α calculated, one may obtain the non-equilibrium stresses as

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = \beta_\infty^\alpha \tilde{\mathbf{S}}_{\text{iso}}^\infty - \frac{\beta_\infty^\alpha}{\mu^\alpha(\Theta)} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\infty}{\partial \tilde{\mathbf{C}}} : \left(\beta_\infty^\alpha \tilde{\mathbf{S}}_{\text{iso}}^\infty - \hat{\mathbf{S}}_0^\alpha - \mathbf{Q}^\alpha \right), \quad (2.48)$$

with $\hat{\mathbf{S}}_0^\alpha = \beta_\infty^\alpha \tilde{\mathbf{S}}_{\text{iso}0}^\infty - \mathbf{Q}_0^\alpha$ and $\tilde{\mathbf{S}}_{\text{iso}0}^\infty := \tilde{\mathbf{S}}_{\text{iso}}^\infty|_{t=0}$. For convenience, we introduce the fictitious elasticity tensor $\tilde{\mathbf{C}}_{\text{iso}}^\infty$ defined as

$$\tilde{\mathbf{C}}_{\text{iso}}^\infty := 2J^{-\frac{4}{3}} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\infty}{\partial \tilde{\mathbf{C}}} = 4J^{-\frac{4}{3}} \frac{\partial^2 G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}}, \quad (2.49)$$

with which the non-equilibrium stresses (2.48) can be expressed as

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = \beta_\infty^\alpha \tilde{\mathbf{S}}_{\text{iso}}^\infty - \frac{\beta_\infty^\alpha}{2\mu^\alpha(\Theta)} J^{\frac{4}{3}} \tilde{\mathbf{C}}_{\text{iso}}^\infty : \left(\beta_\infty^\alpha \tilde{\mathbf{S}}_{\text{iso}}^\infty - \hat{\mathbf{S}}_0^\alpha - \mathbf{Q}^\alpha \right).$$

As will be revealed in Section 2.4.3, in the thermodynamic equilibrium limit, the non-equilibrium stresses given by this model will *not* vanish in general. This behavior is counter-intuitive as the stress from the dissipative potential is expected to be fully relaxed in the limit from the material modeling perspective. In our numerical experiences, a non-vanishing $\tilde{\mathbf{S}}_{\text{neq}}^\alpha$ often leads the body to deform to an unexpected state and sometimes gives unstable material behavior (see Figures 6 and 7). It is therefore necessary to discuss the condition characterizing the relaxation of the non-equilibrium stresses.

2.4.3. Condition for the vanishment of the non-equilibrium stress $\mathbf{S}_{\text{neq}}^\alpha$ in the thermodynamic equilibrium limit

Intuitively, the non-equilibrium stress $\mathbf{S}_{\text{neq}}^\alpha$ (2.40) shall vanish in the thermodynamic equilibrium limit. This property can be conveniently achieved in the multiplicative viscoelasticity theory [26]. Without having the multiplicative decomposition of the deformation gradient, we need to characterize this property by analyzing necessary and sufficient conditions for the vanishment of $\mathbf{S}_{\text{neq}}^\alpha$ in the equilibrium limit. First, the relaxation of the non-equilibrium stresses in the limit is given by

$$\mathbf{O} = \mathbf{S}_{\text{neq}}^\alpha \Big|_{\text{eq}} = \mathbb{P} : \tilde{\mathbf{S}}_{\text{neq}}^\alpha \Big|_{\text{eq}}.$$

It can be proved that the above relation is equivalent to the vanishment of the fictitious stresses $\tilde{\mathbf{S}}_{\text{neq}}^\alpha$,

$$\mathbf{O} = \tilde{\mathbf{S}}_{\text{neq}}^\alpha \Big|_{\text{eq}},$$

with the proof detailed in Appendix A. Recall that the equilibrium [47] is characterized by (2.27) and (2.28), namely,

$$\mathbf{Q}^\alpha \Big|_{\text{eq}} = \mathbf{O}, \quad \text{or equivalently} \quad \frac{d}{dt} \Gamma^\alpha \Big|_{\text{eq}} = \mathbf{O}, \quad \text{for } \alpha = 1, \dots, m.$$

For finite linear viscoelastic materials, the above conditions imply that

$$\mathbf{O} = \mathbf{Q}^\alpha \Big|_{\text{eq}} = \tilde{\mathbf{S}}_{\text{iso}}^\alpha \Big|_{\text{eq}} - \hat{\mathbf{S}}_0^\alpha - 2\mu^\alpha(\Theta) \frac{\Gamma^\alpha - \mathbf{I}}{2} \Big|_{\text{eq}},$$

which is equivalent to

$$\frac{\Gamma^\alpha - \mathbf{I}}{2} \Big|_{\text{eq}} = \frac{1}{2\mu^\alpha(\Theta)} \left(\tilde{\mathbf{S}}_{\text{iso}}^\alpha \Big|_{\text{eq}} - \hat{\mathbf{S}}_0^\alpha \right).$$

From (2.40) and the above relation, we know that in the equilibrium limit,

$$\begin{aligned} \tilde{\mathbf{S}}_{\text{neq}}^\alpha \Big|_{\text{eq}} &= 2 \left(\frac{\partial F^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} \Big|_{\text{eq}} - \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} \Big|_{\text{eq}} : \frac{\Gamma^\alpha - \mathbf{I}}{2} \Big|_{\text{eq}} \right) \\ &= 2 \left(\frac{\partial F^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} \Big|_{\text{eq}} - \frac{1}{2\mu^\alpha(\Theta)} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} \Big|_{\text{eq}} : \left(\tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right) \Big|_{\text{eq}} \right) \\ &= 2 \frac{\partial}{\partial \tilde{\mathbf{C}}} \left(F^\alpha(\tilde{\mathbf{C}}, \Theta) - \frac{1}{4\mu^\alpha(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right|^2 \right) \Big|_{\text{eq}}. \end{aligned}$$

The last equality in the above derivation makes use of the symmetry property of $\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha / \partial \tilde{\mathbf{C}}$. Therefore, the *necessary* condition for ensuring $\tilde{\mathbf{S}}_{\text{neq}}^\alpha \Big|_{\text{eq}} = \mathbf{O}$ is

$$\mathbf{O} = \frac{\partial}{\partial \tilde{\mathbf{C}}} \left(F^\alpha(\tilde{\mathbf{C}}, \Theta) - \frac{1}{4\mu^\alpha(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right|^2 \right) \Big|_{\text{eq}},$$

or equivalently, after a partial integration with respect to $\tilde{\mathbf{C}}$,

$$F^\alpha(\tilde{\mathbf{C}}, \Theta) \Big|_{\text{eq}} = \left(\frac{1}{4\mu^\alpha(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right|^2 + T^\alpha(\Theta) \right) \Big|_{\text{eq}},$$

wherein T^α is an arbitrary function of the temperature. The configurational free energy Υ^α (2.37) for finite linear viscoelasticity in the equilibrium limit is

$$\Upsilon^\alpha \left(\tilde{\mathbf{C}}, \Theta, \Gamma^1, \dots, \Gamma^m \right) \Big|_{\text{eq}} = H^\alpha(\Gamma^\alpha, \Theta) \Big|_{\text{eq}} + \left(\hat{\mathbf{S}}_0^\alpha - \tilde{\mathbf{S}}_{\text{iso}}^\alpha \right) : \frac{\Gamma^\alpha - \mathbf{I}}{2} \Big|_{\text{eq}} + F^\alpha(\tilde{\mathbf{C}}, \Theta) \Big|_{\text{eq}}$$

$$\begin{aligned}
&= \left(\mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2 - 2\mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2 \right) \Big|_{\text{eq}} \\
&\quad + \left(\frac{1}{4\mu^\alpha(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right|^2 + T^\alpha(\Theta) \right) \Big|_{\text{eq}} \\
&= -\mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2 \Big|_{\text{eq}} + \mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2 \Big|_{\text{eq}} + T^\alpha(\Theta) \Big|_{\text{eq}} \\
&= T^\alpha(\Theta) \Big|_{\text{eq}}. \tag{2.50}
\end{aligned}$$

This suggests that, for the finite linear viscoelastic model considered, the mechanical part of the configurational free energy will eventually vanish in the thermodynamic equilibrium limit, if we demand the non-equilibrium stresses to vanish in the limit. Based on the above discussion, a convenient modeling choice can be made by demanding that F^α always satisfies the following relation,

$$F^\alpha(\tilde{\mathbf{C}}, \Theta) = \frac{1}{4\mu^\alpha(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right|^2 + T^\alpha(\Theta), \tag{2.51}$$

from which we readily have

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = 2 \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \left(\frac{1}{2\mu^\alpha(\Theta)} \left(\tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right) - \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right) = \frac{1}{\mu^\alpha(\Theta)} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \mathbf{Q}^\alpha. \tag{2.52}$$

Therefore, the non-equilibrium stresses $\mathbf{S}_{\text{neq}}^\alpha$ automatically vanish when reaching the equilibrium limit, and the relation (2.51) serves as a *sufficient* condition for the configurational free energy that guarantees the stress relaxation. In this work, we adopt (2.51) for the analytic form of F^α , thereby leaving G^α as the only undetermined part of the configurational free energy for modelers.

Remark 7. *In the literature, the configurational free energy is often proposed with $F^\alpha = G^\alpha$ in (2.37) [47, 51]. Based on the above analysis, the non-equilibrium stress arising from that configurational free energy will not vanish in the thermodynamic equilibrium limit, except for one special case to be discussed in the next section.*

2.4.4. The Holzapfel-Simo-Saint Venant-Kirchhoff model

For finite linear viscoelastic materials that satisfy the condition (2.51), the stresses are completely determined in terms of G^α . Here, as an illustrative example, we consider a special form of G^α ,

$$G^\alpha(\tilde{\mathbf{C}}, \Theta) = \mu^\alpha(\Theta) \left| \frac{\tilde{\mathbf{C}} - \mathbf{I}}{2} \right|^2. \tag{2.53}$$

Based on (2.38), one readily has

$$\tilde{\mathbf{S}}_{\text{iso}}^\alpha = 2 \frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} = \mu^\alpha(\Theta) (\tilde{\mathbf{C}} - \mathbf{I}), \text{ and } \mathbf{Q}^\alpha = \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha - \mu^\alpha(\Theta) (\mathbf{\Gamma}^\alpha - \mathbf{I}) = \mu^\alpha(\Theta) (\tilde{\mathbf{C}} - \mathbf{\Gamma}^\alpha) + \mathbf{Q}_0^\alpha.$$

The evolution equation (2.43) can be written as

$$\begin{aligned}
\frac{d}{dt} \mathbf{Q}^\alpha + \frac{\mathbf{Q}^\alpha}{\tau^\alpha} &= \frac{d}{dt} \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \frac{d\mu^\alpha(\Theta)}{dt} (\mathbf{\Gamma}^\alpha - \mathbf{I}) = \mu^\alpha(\Theta) \frac{d}{dt} \tilde{\mathbf{C}} + \frac{d\mu^\alpha(\Theta)}{dt} (\tilde{\mathbf{C}} - \mathbf{I}) - \frac{d\mu^\alpha(\Theta)}{dt} (\mathbf{\Gamma}^\alpha - \mathbf{I}) \\
&= \mu^\alpha(\Theta) \frac{d}{dt} \tilde{\mathbf{C}} + \frac{d\mu^\alpha(\Theta)}{dt} (\tilde{\mathbf{C}} - \mathbf{\Gamma}^\alpha).
\end{aligned}$$

The fictitious non-equilibrium stress $\tilde{\mathbf{S}}_{\text{neq}}^\alpha$ can be represented as

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = \frac{1}{\mu^\alpha(\Theta)} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \mathbf{Q}^\alpha = \frac{1}{\mu^\alpha(\Theta)} \mu^\alpha(\Theta) \mathbf{Q}^\alpha = \mathbf{Q}^\alpha.$$

Consequently, we have the fictitious second Piola-Kirchhoff stress represented as

$$\tilde{\mathbf{S}} = \tilde{\mathbf{S}}_{\text{iso}}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbf{S}}_{\text{neq}}^{\alpha} = \tilde{\mathbf{S}}_{\text{iso}}^{\infty} + \sum_{\alpha=1}^m \mathbf{Q}^{\alpha}.$$

This recovers one specific model proposed by Holzapfel and Simo in [51, Section 4.2], which can be viewed as a generalization of the Saint Venant-Kirchhoff model to the viscous regime. This is the reason for its name used here. We should note that directly using \mathbf{Q}^{α} in place of $\tilde{\mathbf{S}}_{\text{neq}}^{\alpha}$ is fairly common in the literature [38, 42, 47, 49], and the above analysis shows that this choice implicitly implies a special form of the configurational free energy. We note that, to ensure $\tilde{\mathbf{S}}_{\text{neq}}^{\alpha} = \mathbf{Q}^{\alpha}$, one needs

$$\frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^{\alpha}}{\partial \tilde{\mathbf{C}}} = 2 \frac{\partial^2 G^{\alpha}(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} = \mu^{\alpha}(\Theta) \mathbb{I},$$

which implies a quadratic form of G^{α} in terms of $\tilde{\mathbf{C}}$. Therefore, enforcing $\tilde{\mathbf{S}}_{\text{neq}}^{\alpha} = \mathbf{Q}^{\alpha}$ implicitly requires G^{α} to be quadratic in terms of $\tilde{\mathbf{C}}$. This also suggests a linear relationship between $\tilde{\mathbf{S}}_{\text{iso}}^{\alpha}$ and $\tilde{\mathbf{C}}$. For general nonlinear models, apparently one should distinguish \mathbf{Q}^{α} from $\tilde{\mathbf{S}}_{\text{neq}}^{\alpha}$. Furthermore, according to (2.51), we have

$$F^{\alpha}(\tilde{\mathbf{C}}, \Theta) = \frac{1}{4\mu^{\alpha}(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^{\alpha} - \hat{\mathbf{S}}_0^{\alpha} \right|^2 + T^{\alpha}(\Theta) = \frac{1}{4\mu^{\alpha}(\Theta)} \left| \mu^{\alpha}(\Theta) (\tilde{\mathbf{C}} - \mathbf{I}) - \hat{\mathbf{S}}_0^{\alpha} \right|^2 + T^{\alpha}(\Theta).$$

The configurational free energy (2.37) can be written as

$$\begin{aligned} \Upsilon^{\alpha}(\tilde{\mathbf{C}}, \Theta, \Gamma^1, \dots, \Gamma^m) &= H^{\alpha}(\Gamma^{\alpha}, \Theta) + \left(\hat{\mathbf{S}}_0^{\alpha} - 2 \frac{\partial G^{\alpha}(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} \right) : \frac{\Gamma^{\alpha} - \mathbf{I}}{2} + F^{\alpha}(\tilde{\mathbf{C}}, \Theta) \\ &= \mu^{\alpha}(\Theta) \left| \frac{\Gamma^{\alpha} - \mathbf{I}}{2} \right|^2 + \left(\hat{\mathbf{S}}_0^{\alpha} - 2\mu^{\alpha}(\Theta) \frac{\tilde{\mathbf{C}} - \mathbf{I}}{2} \right) : \frac{\Gamma^{\alpha} - \mathbf{I}}{2} \\ &\quad + \frac{1}{4\mu^{\alpha}(\Theta)} \left| \mu^{\alpha}(\Theta) (\tilde{\mathbf{C}} - \mathbf{I}) - \hat{\mathbf{S}}_0^{\alpha} \right|^2 + T^{\alpha}(\Theta) \\ &= \mu^{\alpha}(\Theta) \left| \frac{\tilde{\mathbf{C}} - \Gamma^{\alpha}}{2} \right|^2 - \hat{\mathbf{S}}_0^{\alpha} : \frac{\tilde{\mathbf{C}} - \Gamma^{\alpha}}{2} + \frac{1}{4\mu^{\alpha}(\Theta)} \left| \hat{\mathbf{S}}_0^{\alpha} \right|^2 + T^{\alpha}(\Theta) \\ &= \frac{1}{4\mu^{\alpha}(\Theta)} \left| \mu^{\alpha}(\Theta) (\tilde{\mathbf{C}} - \Gamma^{\alpha}) - \hat{\mathbf{S}}_0^{\alpha} \right|^2 + T^{\alpha}(\Theta). \end{aligned} \quad (2.54)$$

One can see that for the model considered above, the configurational free energy is a function of $(\tilde{\mathbf{C}} - \Gamma^{\alpha})/2$, which can be interpreted as the elastic strain measure in the standard linear solid model [49, p. 286], under an additive split of the Green-Lagrange strain. Although is less frequently used in inelasticity when compared with the multiplicative decomposition [26, 97], the additive split has been shown to be supported by thermodynamic principles [57, 58] and has been utilized to model elastoplasticity [98]. We may regard the Holzapfel-Simo-Saint Venant-Kirchhoff model as a viscoelasticity theory based on a similar additive strain split concept (see also [99]), in contrast to the multiplicative viscoelasticity theory [26, 35, 36, 100, 101].

Remark 8. *It is convenient to make a rheological interpretation of this model. It can be seen that in this model $\tilde{\mathbf{S}}_{\text{iso},0}^{\alpha} = \mathbf{O}$ as $\tilde{\mathbf{C}}|_{t=0} = \mathbf{I}$. If we choose $\mathbf{Q}_0^{\alpha} = \mathbf{O}$, then $\hat{\mathbf{S}}_0^{\alpha} = \mathbf{O}$ according to (2.45). We may introduce an *infinitesimal* strain ε as an approximation of $(\tilde{\mathbf{C}} - \mathbf{I})/2$,*

$$\varepsilon \approx \frac{\tilde{\mathbf{C}} - \mathbf{I}}{2}.$$

Similarly, there is a small-strain approximation of $(\mathbf{\Gamma}^\alpha - \mathbf{I})/2$,

$$\boldsymbol{\varepsilon}_v^\alpha \approx \frac{\tilde{\mathbf{\Gamma}}^\alpha - \mathbf{I}}{2}.$$

If we denote $\boldsymbol{\varepsilon}_e^\alpha := \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_v^\alpha$, it can be shown that under the infinitesimal strain approximation, the configurational free energy (2.54) can be approximated by

$$\Upsilon^\alpha \approx \mu^\alpha(\Theta) |\boldsymbol{\varepsilon}_e^\alpha|^2 + T^\alpha(\Theta).$$

This can be interpreted as the elastic energy of the springs within the Maxwell elements of the generalized Zener model.

2.4.5. A modified model based on the identical-polymer-chain assumption

From the analysis in Section 2.4.3, we can see that the identical polymer chain model stated in Section 2.4.2 will engender counter-intuitive behavior of the non-equilibrium stress. In this section, we revisit the identical polymer chain model, by adopting the form (2.51) for F^α . The remaining component for the configurational free energy is given by

$$G^\alpha(\tilde{\mathbf{C}}, \Theta) = \beta_\alpha^\infty G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta), \quad (2.55)$$

with $\beta_\alpha^\infty \in (0, \infty)$ being non-dimensional constants [37, 41]. With this modeling assumption, the material behavior is completely characterized by the form of G_{iso}^∞ , and we have

$$\tilde{\mathbf{S}}_{\text{iso}}^\alpha = 2 \frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} = \beta_\alpha^\infty 2 \frac{\partial G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} = \beta_\alpha^\infty \tilde{\mathbf{S}}_{\text{iso}}^\infty.$$

With this elasticity tensor, the non-equilibrium stress (2.52) can be represented as

$$\tilde{\mathbf{S}}_{\text{neq}}^\alpha = \frac{1}{\mu^\alpha} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \mathbf{Q}^\alpha = \frac{J^{\frac{4}{3}} \beta_\alpha^\infty}{2\mu^\alpha} \tilde{\mathbf{C}}_{\text{iso}}^\infty : \mathbf{Q}^\alpha = \frac{2\beta_\alpha^\infty}{\mu^\alpha} \frac{\partial^2 G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} : \mathbf{Q}^\alpha.$$

Remark 9. In the literature, the terminology “ground-stress” is often used to refer the fictitious stress $\mathbf{S}_{\text{iso}}^\infty$ [102, 103]; “over-stress” is used to refer \mathbf{Q}^α [26, 40] or $\mathbf{S}_{\text{neq}}^\alpha$ [103]. The prevailing custom is to use \mathbf{Q}^α and $\tilde{\mathbf{S}}_{\text{neq}}^\alpha$ interchangeably [37, 41, 49]. However, the above analysis indicates that, for a general nonlinear form of G^α like (2.55), $\tilde{\mathbf{S}}_{\text{neq}}^\alpha \neq \mathbf{Q}^\alpha$.

Remark 10. Interestingly, if the functional form of G_{iso}^∞ is given by the neo-Hookean model, we have

$$\frac{\partial^2 G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} = \mathbb{O},$$

which indicates that $\mathbf{S}_{\text{neq}}^\alpha = \mathbf{O}$. In other words, under the identical-polymer-chain assumption, the neo-Hookean material does not present viscous stress in finite linear viscoelasticity.

We may summarize the thermodynamic potential and the resulting constitutive relations for finite linear thermal-visco-hyperelastic materials discussed in Section 2 as follows.

Constitutive laws for finite linear thermo-visco-elasticity

Gibbs free energy:

$$G(\tilde{\mathbf{C}}, P, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = G_{\text{vol}}^\infty(P, \Theta) + G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta) + \sum_{\alpha=1}^m \Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m),$$

$$\Upsilon^\alpha(\tilde{\mathbf{C}}, \Theta, \mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^m) = H^\alpha(\mathbf{\Gamma}^\alpha, \Theta) + \left(\hat{\mathbf{S}}_0^\alpha - 2 \frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} \right) : \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} + F^\alpha(\tilde{\mathbf{C}}, \Theta),$$

$$H^\alpha(\mathbf{\Gamma}^\alpha, \Theta) = \mu^\alpha(\Theta) \left| \frac{\mathbf{\Gamma}^\alpha - \mathbf{I}}{2} \right|^2, \quad F^\alpha(\tilde{\mathbf{C}}, \Theta) = \frac{1}{4\mu^\alpha(\Theta)} \left| \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \hat{\mathbf{S}}_0^\alpha \right|^2 + T^\alpha(\Theta).$$

Second Piola-Kirchhoff stress:

$$\mathbf{S} = \mathbf{S}_{\text{iso}} + \mathbf{S}_{\text{vol}}, \quad \mathbf{S}_{\text{iso}} = J^{-\frac{2}{3}} \mathbb{P} : \tilde{\mathbf{S}}, \quad \mathbf{S}_{\text{vol}} = -JPC^{-1}, \quad \tilde{\mathbf{S}} = \tilde{\mathbf{S}}_{\text{iso}}^\infty + \sum_{\alpha=1}^m \tilde{\mathbf{S}}_{\text{neq}}^\alpha,$$

$$\tilde{\mathbf{S}}_{\text{iso}}^\infty = 2 \frac{\partial G_{\text{iso}}^\infty}{\partial \tilde{\mathbf{C}}}, \quad \tilde{\mathbf{S}}_{\text{neq}}^\alpha = \frac{1}{\mu^\alpha(\Theta)} \frac{\partial \tilde{\mathbf{S}}_{\text{iso}}^\alpha}{\partial \tilde{\mathbf{C}}} : \mathbf{Q}^\alpha, \quad \tilde{\mathbf{S}}_{\text{iso}}^\alpha = 2 \frac{\partial G^\alpha}{\partial \tilde{\mathbf{C}}}.$$

Cauchy stress:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{dev}} - p\mathbf{I}, \quad \boldsymbol{\sigma}_{\text{dev}} = \frac{1}{J} \mathbf{F} \mathbf{S}_{\text{iso}} \mathbf{F}^T.$$

Entropy:

$$\mathfrak{E} = \mathfrak{E}^\infty + \sum_{\alpha=1}^m \mathfrak{E}^\alpha, \quad \mathfrak{E}^\infty := -\frac{\partial G_{\text{vol}}^\infty}{\partial \Theta} - \frac{\partial G_{\text{iso}}^\infty}{\partial \Theta}, \quad \mathfrak{E}^\alpha := -\frac{\partial \Upsilon^\alpha}{\partial \Theta}.$$

Dissipation:

$$\mathcal{D} = \frac{\bar{\kappa}}{\Theta} |\nabla_{\mathbf{x}} \Theta|^2 + \sum_{\alpha=1}^m \frac{\eta^\alpha}{2\Theta} \left| \frac{d}{dt} \mathbf{\Gamma}^\alpha \right|^2 = \frac{\bar{\kappa}}{\Theta} |\nabla_{\mathbf{x}} \Theta|^2 + \sum_{\alpha=1}^m \frac{1}{2\Theta \eta^\alpha} |\mathbf{Q}^\alpha|^2.$$

Constitutive relations for \mathbf{Q}^α :

$$\mathbf{Q}^\alpha = \eta^\alpha \frac{d}{dt} \mathbf{\Gamma}^\alpha.$$

Evolution equations for \mathbf{Q}^α :

$$\begin{cases} \frac{d}{dt} \mathbf{Q}^\alpha + \frac{1}{\tau^\alpha} \mathbf{Q}^\alpha = \frac{d}{dt} \tilde{\mathbf{S}}_{\text{iso}}^\alpha - \frac{d\mu^\alpha(\Theta)}{dt} (\mathbf{\Gamma}^\alpha - \mathbf{I}), \\ \mathbf{Q}^\alpha|_{t=0} = \mathbf{Q}_0^\alpha. \end{cases}$$

Hereditary integral form for \mathbf{Q}^α :

$$\mathbf{Q}^\alpha = \exp(-t/\tau^\alpha) \mathbf{Q}_0^\alpha + \int_{0+}^t \exp(-(t-s)/\tau^\alpha) \left(\frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^\alpha - (\mathbf{\Gamma}^\alpha - \mathbf{I}) \frac{d\mu^\alpha(\Theta)}{ds} \right) ds.$$

3. Numerical formulation

In this section, we develop a suite of discretization methods for the material model developed in Section 2 by restricting our discussion to *fully incompressible* materials under the *isothermal* condition. We will

prove that the proposed numerical formulation is stable in energy. Henceforth, we use the acronyms IPC, HS, and MIPC models to stand for the identical polymer chain model stated in Section 2.4.2, the Holzapfel-Simo-Saint Venant-Kirchhoff model stated in Section 2.4.4, and the modified identical polymer chain model stated in Section 2.4.5, respectively.

3.1. Strong-form problem

Under isothermal conditions, the energy equation is decoupled from the mechanical system, and the motion of an incompressible continuum body is governed by the following system of equations,

$$\mathbf{0} = \frac{d\mathbf{u}}{dt} - \mathbf{v}, \quad \text{in } \Omega_{\mathbf{x}}^t, \quad (3.1)$$

$$0 = \nabla_{\mathbf{x}} \cdot \mathbf{v}, \quad \text{in } \Omega_{\mathbf{x}}^t, \quad (3.2)$$

$$\mathbf{0} = \rho(J) \frac{d\mathbf{v}}{dt} - \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}_{\text{dev}} + \nabla_{\mathbf{x}} p - \rho(J) \mathbf{b}, \quad \text{in } \Omega_{\mathbf{x}}^t. \quad (3.3)$$

The boundary $\Gamma_{\mathbf{x}}^t := \partial\Omega_{\mathbf{x}}^t$ enjoys a non-overlapping subdivision: $\Gamma_{\mathbf{x}}^t = \Gamma_{\mathbf{x}}^{g,t} \cup \Gamma_{\mathbf{x}}^{h,t}$, wherein $\Gamma_{\mathbf{x}}^{g,t}$ and $\Gamma_{\mathbf{x}}^{h,t}$ are the Dirichlet and Neumann parts of the boundary, respectively. The boundary conditions can be stated as

$$\mathbf{u} = \mathbf{g}, \text{ on } \Gamma_{\mathbf{x}}^{g,t}, \quad \mathbf{v} = \frac{d\mathbf{g}}{dt}, \text{ on } \Gamma_{\mathbf{x}}^{g,t}, \quad (\boldsymbol{\sigma}_{\text{dev}} - p\mathbf{I})\mathbf{n} = \mathbf{h}, \text{ on } \Gamma_{\mathbf{x}}^{h,t}. \quad (3.4)$$

Given the initial data \mathbf{u}_0 , p_0 , and \mathbf{v}_0 , the initial-boundary value problem is to solve for \mathbf{u} , p , and \mathbf{v} that satisfy (3.1)-(3.4), and

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad p(\mathbf{x}, 0) = p_0(\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}). \quad (3.5)$$

Equations (3.1)-(3.5) constitute an initial-boundary value problem. The constitutive relations for the material have been given in Section 2. For fully incompressible materials, it can be shown that the volumetric part of the free energy adopts the form $G_{\text{vol}}^{\infty} = P$ [60], which leads to $\rho = \rho_0$ from (2.22). In the formulation, we adopt a modified constitutive relation for the density, that is $\rho(J) = \rho_0/J$ [65]. Apparently, at the continuum level, this relation is equivalent to $\rho(J) = \rho_0$, as the divergence-free condition for the velocity guarantees $J = 1$. Yet, we note that at the discrete level, the condition $J = 1$ in general rarely holds in the pointwise sense [59]. The adoption of the relation $\rho(J) = \rho_0/J$ facilitates our discussion of the nonlinear numerical stability, as will be shown in the next section.

3.2. Semi-discrete formulation

Consider a unite cube $\hat{\Omega} := (0, 1)^3$, which is typically referred to as the parametric domain. Given open knot vectors Ξ_l for $l = 1, 2, 3$, a parametric Cartesian mesh can be constructed that partitions $\hat{\Omega}$ into rectangular parallelepipeds. If a set of weights is also given, the NURBS basis function can be constructed, and we use the notation $\mathcal{R}_{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3}^{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3}$ to represent the space of multivariate NURBS basis functions of degree p_l and interelement regularity vector \mathbf{r}_l in the l -th parametric direction. The values of p_l and \mathbf{r}_l are completely determined by the knot vector Ξ_l . Readers are referred to [71] for more technical details about the spline technologies. We may then define two discrete function spaces on $\hat{\Omega}$,

$$\hat{\mathcal{S}}_h := \mathcal{R}_{\mathbf{r}_1+\mathbf{b}, \mathbf{r}_2+\mathbf{b}, \mathbf{r}_3+\mathbf{b}}^{\mathbf{p}+\mathbf{a}, \mathbf{p}+\mathbf{a}, \mathbf{p}+\mathbf{a}} \times \mathcal{R}_{\mathbf{r}_1+\mathbf{b}, \mathbf{r}_2+\mathbf{b}, \mathbf{r}_3+\mathbf{b}}^{\mathbf{p}+\mathbf{a}, \mathbf{p}+\mathbf{a}, \mathbf{p}+\mathbf{a}} \times \mathcal{R}_{\mathbf{r}_1+\mathbf{b}, \mathbf{r}_2+\mathbf{b}, \mathbf{r}_3+\mathbf{b}}^{\mathbf{p}+\mathbf{a}, \mathbf{p}+\mathbf{a}, \mathbf{p}+\mathbf{a}}, \quad \hat{\mathcal{P}}_h := \mathcal{R}_{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3}^{\mathbf{p}, \mathbf{p}, \mathbf{p}},$$

with integer parameters $1 \leq \mathbf{a}$ and $0 \leq \mathbf{b} < \mathbf{a}$. Assuming the reference configuration of the body can be parametrized by a smooth geometrical mapping $\boldsymbol{\psi} : \hat{\Omega} \rightarrow \Omega_{\mathbf{X}}$ with smooth inverse. The boundary of $\Omega_{\mathbf{X}}$ can be partitioned into two non-overlapping subdivisions as $\partial\Omega_{\mathbf{x}} := \Gamma_{\mathbf{X}}^G \cup \Gamma_{\mathbf{X}}^H$, and the two subdivisions satisfy $\Gamma_{\mathbf{X}}^G = \boldsymbol{\varphi}_t^{-1}(\Gamma_{\mathbf{x}}^{g,t})$ and $\Gamma_{\mathbf{X}}^H = \boldsymbol{\varphi}_t^{-1}(\Gamma_{\mathbf{x}}^{h,t})$. The relation between the two configurations and the boundary subdivisions are illustrated in Figure 2. The discrete function spaces on $\Omega_{\mathbf{X}}$ can be defined through the pull-back operation,

$$\mathcal{S}_h := \{\mathbf{w} : \mathbf{w} \circ \boldsymbol{\psi} \in \hat{\mathcal{S}}_h\}, \quad \mathcal{P}_h := \{q : q \circ \boldsymbol{\psi} \in \hat{\mathcal{P}}_h\}.$$

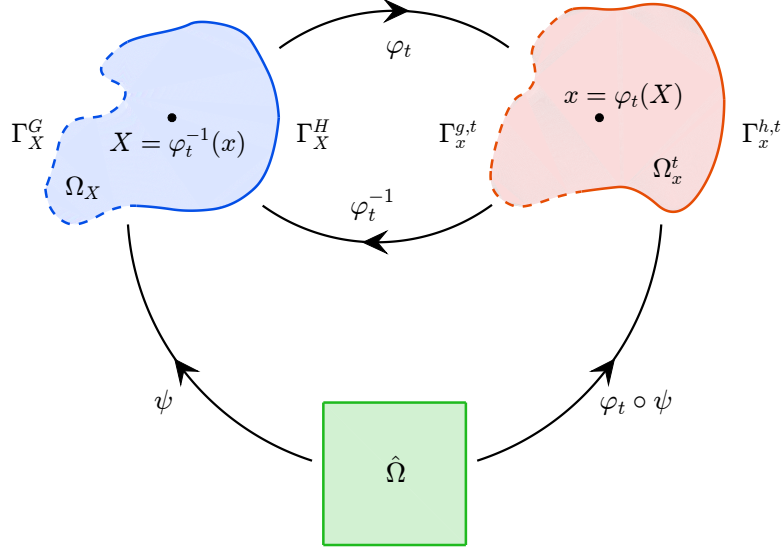


Figure 2: Illustration of the reference and current configurations with the boundary subdivisions. The spline functions are defined on $\hat{\Omega}$ and are pushed forward to the two configurations via the mappings ψ and $\varphi_t \circ \psi$.

In this work, the mapping ψ is constructed from the NURBS basis functions in $\hat{\mathcal{S}}_h$, making \mathcal{S}_h an isoparametric discrete space. With the spaces \mathcal{S}_h and \mathcal{P}_h defined, we may specify the trial solution spaces on the reference configuration as

$$\begin{aligned} \mathcal{S}_{U_h} &= \left\{ \mathbf{U}_h : \mathbf{U}_h(\cdot, t) \in \mathcal{S}_h, \quad \mathbf{U}_h(\cdot, t) = \mathbf{G} \text{ on } \Gamma_{\mathbf{X}}^G, \quad t \in [0, T] \right\}, \\ \mathcal{S}_{P_h} &= \left\{ P_h : P_h(\cdot, t) \in \mathcal{P}_h, \quad t \in [0, T] \right\}, \\ \mathcal{S}_{V_h} &= \left\{ \mathbf{V}_h : \mathbf{V}_h(\cdot, t) \in \mathcal{S}_h, \quad \mathbf{V}_h(\cdot, t) = \frac{d\mathbf{G}}{dt} \text{ on } \Gamma_{\mathbf{X}}^G, \quad t \in [0, T] \right\}, \end{aligned}$$

and the corresponding test function spaces are defined as

$$\mathcal{V}_{P_h} = \mathcal{S}_{P_h}, \quad \mathcal{V}_{V_h} = \left\{ \mathbf{W}_h : \mathbf{W}_h(\cdot, t) \in \mathcal{S}_h, \quad \mathbf{W}_h(\cdot, t) = \mathbf{0} \text{ on } \Gamma_{\mathbf{X}}^G, \quad t \in [0, T] \right\}.$$

Given the displacement \mathbf{U}_h , the placement field is given by $\varphi_h = \mathbf{U}_h + \mathbf{X}$. Consequently, we may also state the trial solution space defined on the current configuration as

$$\begin{aligned} \mathcal{S}_{\mathbf{u}_h} &= \left\{ \mathbf{u}_h : \mathbf{u}_h \circ \varphi_h \in \mathcal{S}_h, \quad \mathbf{u}_h(\cdot, t) = \mathbf{g} \text{ on } \Gamma_{\mathbf{x}}^{g,t}, \quad t \in [0, T] \right\}, \\ \mathcal{S}_{p_h} &= \left\{ p_h : p_h \circ \varphi_h \in \mathcal{P}_h, \quad t \in [0, T] \right\}, \\ \mathcal{S}_{\mathbf{v}_h} &= \left\{ \mathbf{v}_h : \mathbf{v}_h \circ \varphi_h \in \mathcal{S}_h, \quad \mathbf{v}_h(\cdot, t) = \frac{d\mathbf{g}}{dt} \text{ on } \Gamma_{\mathbf{x}}^{g,t}, \quad t \in [0, T] \right\}, \end{aligned}$$

and the test function spaces are defined as

$$\mathcal{V}_{p_h} = \mathcal{S}_{p_h}, \quad \mathcal{V}_{\mathbf{v}_h} = \left\{ \mathbf{w}_h : \mathbf{w}_h \circ \varphi_h \in \mathcal{S}_h, \quad \mathbf{w}_h(\cdot, t) = \mathbf{0} \text{ on } \Gamma_{\mathbf{x}}^{g,t}, \quad t \in [0, T] \right\}.$$

With the discrete function defined above, the semi-discrete formulation on the current configuration can be stated as follows. Find $\mathbf{y}_h(t) := \{\mathbf{u}_h(t), p_h(t), \mathbf{v}_h(t)\}^T \in \mathcal{S}_{\mathbf{u}_h} \times \mathcal{S}_{p_h} \times \mathcal{S}_{\mathbf{v}_h}$ such that for $t \in [0, T]$,

$$\mathbf{0} = \mathbf{B}^k(\dot{\mathbf{y}}_h, \mathbf{y}_h) := \frac{d\mathbf{u}_h}{dt} - \mathbf{v}_h, \quad (3.6)$$

$$0 = \mathbf{B}^p(q_h; \dot{\mathbf{y}}_h, \mathbf{y}_h) := \int_{\Omega_{\mathbf{x}}^t} q_h \nabla_{\mathbf{x}} \cdot \mathbf{v}_h d\Omega_{\mathbf{x}}, \quad (3.7)$$

$$0 = \mathbf{B}^m(\mathbf{w}_h; \dot{\mathbf{y}}_h, \mathbf{y}_h) := \int_{\Omega_{\mathbf{x}}^t} \mathbf{w}_h \cdot \rho(J_h) \frac{d\mathbf{v}_h}{dt} + \nabla_{\mathbf{x}} \mathbf{w}_h : \boldsymbol{\sigma}_{\text{dev}} - \nabla_{\mathbf{x}} \cdot \mathbf{w}_h p_h - \mathbf{w}_h \cdot \rho(J_h) \mathbf{b} d\Omega_{\mathbf{x}} \\ - \int_{\Gamma_{\mathbf{x}}^{h,t}} \mathbf{w}_h \cdot \mathbf{h} d\Gamma_{\mathbf{x}}, \quad (3.8)$$

for $\forall \{q_h, \mathbf{w}_h\} \in \mathcal{V}_{p_h} \times \mathcal{V}_{\mathbf{v}_h}$, with $\mathbf{y}_h(0) := \{\mathbf{u}_{h0}, p_{h0}, \mathbf{v}_{h0}\}^T$. Here \mathbf{u}_{h0} , p_{h0} , and \mathbf{v}_{h0} are the \mathcal{L}^2 projections of the initial data onto the finite dimensional trial solution spaces. Alternatively, the semi-discrete formulation can be pulled back to the reference configuration, which can be stated as follows. Find $\mathbf{Y}_h(t) := \{\mathbf{U}_h(t), P_h(t), \mathbf{V}_h(t)\}^T \in \mathcal{S}_{\mathbf{U}_h} \times \mathcal{S}_{P_h} \times \mathcal{S}_{\mathbf{V}_h}$ such that for $t \in [0, T]$,

$$\mathbf{0} = \mathbf{B}^k(\dot{\mathbf{Y}}_h, \mathbf{Y}_h) := \frac{d\mathbf{U}_h}{dt} - \mathbf{V}_h, \quad (3.9)$$

$$0 = \mathbf{B}^p(Q_h; \dot{\mathbf{Y}}_h, \mathbf{Y}_h) := \int_{\Omega_{\mathbf{X}}} Q_h J_h \nabla_{\mathbf{X}} \mathbf{V}_h : \mathbf{F}_h^{-T} d\Omega_{\mathbf{X}}, \quad (3.10)$$

$$0 = \mathbf{B}^m(\mathbf{W}_h; \dot{\mathbf{Y}}_h, \mathbf{Y}_h) := \int_{\Omega_{\mathbf{X}}} \left(\mathbf{W}_h \cdot \rho_0 \frac{d\mathbf{V}_h}{dt} + \nabla_{\mathbf{X}} \mathbf{W}_h : (J_h \boldsymbol{\sigma}_{\text{dev}} \mathbf{F}_h^{-T}) - J_h P_h \nabla_{\mathbf{X}} \mathbf{W}_h : \mathbf{F}_h^{-T} \right. \\ \left. - \mathbf{W}_h \cdot \rho_0 \mathbf{B} \right) d\Omega_{\mathbf{X}} - \int_{\Gamma_{\mathbf{X}}^H} \mathbf{W}_h \cdot \mathbf{H} d\Gamma_{\mathbf{X}}, \quad (3.11)$$

for $\forall \{Q_h, \mathbf{W}_h\} \in \mathcal{V}_{P_h} \times \mathcal{V}_{\mathbf{V}_h}$, with $\mathbf{Y}_h(0) = \{\mathbf{U}_{h0}, P_{h0}, \mathbf{V}_{h0}\}^T$. Here the initial data are related by $\mathbf{U}_{h0} = \mathbf{u}_{h0} \circ \boldsymbol{\varphi}_t$, $P_{h0} = p_{h0} \circ \boldsymbol{\varphi}_t$, and $\mathbf{V}_{h0} = \mathbf{v}_{h0} \circ \boldsymbol{\varphi}_t$. It can be shown that the above semi-discrete formulation inherits the dissipation property from the continuum model from the following theorem.

Theorem 1 (A priori energy stability). *Assuming the Dirichlet boundary data \mathbf{G} is time independent, the solutions of the semi-discrete problem (3.9)-(3.11) satisfy*

$$\frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \frac{1}{2} \rho_0 \|\mathbf{V}_h\|^2 + G_{\text{iso}}^{\infty}(\tilde{\mathcal{C}}_h) + \sum_{\alpha=1}^m \Upsilon^{\alpha}(\tilde{\mathcal{C}}_h, \Gamma_h^{\alpha}) d\Omega_{\mathbf{X}} = \int_{\Omega_{\mathbf{X}}} \rho_0 \mathbf{V}_h \cdot \mathbf{B} d\Omega_{\mathbf{X}} + \int_{\Gamma_{\mathbf{X}}} \mathbf{V}_h \cdot \mathbf{H} d\Gamma_{\mathbf{X}} \\ - \sum_{\alpha=1}^m \int_{\Omega_{\mathbf{X}}} \frac{1}{4} \left(\frac{d}{dt} \Gamma_h^{\alpha} \right) : \mathbb{V}^{\alpha} : \left(\frac{d}{dt} \Gamma_h^{\alpha} \right) d\Omega_{\mathbf{X}}. \quad (3.12)$$

Proof. Since the Dirichlet boundary data \mathbf{G} is independent of time, one is allowed to choose $Q_h = P_h$ and $\mathbf{W}_h = \mathbf{V}_h$, which leads to

$$0 = \mathbf{B}^p(P_h; \dot{\mathbf{Y}}_h, \mathbf{Y}_h) + \mathbf{B}^m(\mathbf{V}_h; \dot{\mathbf{Y}}_h, \mathbf{Y}_h) \\ = \int_{\Omega_{\mathbf{X}}} P_h J_h \nabla_{\mathbf{X}} \mathbf{V}_h : \mathbf{F}_h^{-T} d\Omega_{\mathbf{X}} + \int_{\Omega_{\mathbf{X}}} \mathbf{V}_h \cdot \rho_0 \frac{d\mathbf{V}_h}{dt} + \nabla_{\mathbf{X}} \mathbf{V}_h : (J_h \boldsymbol{\sigma}_{\text{dev}} \mathbf{F}_h^{-T}) - J_h P_h \nabla_{\mathbf{X}} \mathbf{V}_h : \mathbf{F}_h^{-T} \\ - \mathbf{V}_h \cdot \rho_0 \mathbf{B} d\Omega_{\mathbf{x}} - \int_{\Gamma_{\mathbf{X}}^H} \mathbf{V}_h \cdot \mathbf{H} d\Gamma_{\mathbf{x}} \\ = \frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \frac{1}{2} \rho_0 \|\mathbf{V}_h\|^2 d\Omega_{\mathbf{X}} + \int_{\Omega_{\mathbf{X}}} \frac{d}{dt} \mathbf{F}_h : \frac{\partial \left(G_{\text{iso}}^{\infty}(\tilde{\mathcal{C}}_h) + \sum_{\alpha=1}^m \Upsilon^{\alpha}(\tilde{\mathcal{C}}_h, \Gamma_h^{\alpha}) \right)}{\partial \mathbf{F}_h} - \mathbf{V}_h \cdot \rho_0 \mathbf{B} d\Omega_{\mathbf{X}} \\ - \int_{\Gamma_{\mathbf{X}}} \mathbf{V}_h \cdot \mathbf{H} d\Gamma_{\mathbf{X}} \\ = \frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \frac{1}{2} \rho_0 \|\mathbf{V}_h\|^2 d\Omega_{\mathbf{X}} + \frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \left(G_{\text{iso}}^{\infty}(\tilde{\mathcal{C}}_h) + \sum_{\alpha=1}^m \Upsilon^{\alpha}(\tilde{\mathcal{C}}_h, \Gamma_h^{\alpha}) \right) d\Omega_{\mathbf{X}} + \int_{\Omega_{\mathbf{X}}} \sum_{\alpha=1}^m \mathbf{Q}_h^{\alpha} : \frac{1}{2} \frac{d}{dt} \Gamma_h^{\alpha}$$

$$\begin{aligned}
& - \mathbf{V}_h \cdot \rho_0 \mathbf{B} d\Omega_{\mathbf{X}} - \int_{\Gamma_{\mathbf{X}}} \mathbf{V}_h \cdot \mathbf{H} d\Gamma_{\mathbf{X}} \\
& = \frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \frac{1}{2} \rho_0 \|\mathbf{V}_h\|^2 + G_{\text{iso}}^\infty(\tilde{\mathbf{C}}_h) + \sum_{\alpha=1}^m \Upsilon^\alpha(\tilde{\mathbf{C}}_h, \mathbf{\Gamma}_h^\alpha) d\Omega_{\mathbf{X}} + \int_{\Omega_{\mathbf{X}}} \sum_{\alpha=1}^m \frac{1}{4} \left(\frac{d}{dt} \mathbf{\Gamma}_h^\alpha \right) : \mathbb{V}^\alpha : \left(\frac{d}{dt} \mathbf{\Gamma}_h^\alpha \right) \\
& - \mathbf{V}_h \cdot \rho_0 \mathbf{B} d\Omega_{\mathbf{X}} - \int_{\Gamma_{\mathbf{X}}} \mathbf{V}_h \cdot \mathbf{H} d\Gamma_{\mathbf{X}}.
\end{aligned}$$

Rearranging terms in the above equality results in

$$\frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \frac{1}{2} \rho_0 \|\mathbf{V}_h\|^2 + G_{\text{iso}}^\infty(\tilde{\mathbf{C}}_h) + \sum_{\alpha=1}^m \Upsilon^\alpha(\tilde{\mathbf{C}}_h, \mathbf{\Gamma}_h^\alpha) d\Omega_{\mathbf{X}} = \int_{\Omega_{\mathbf{X}}} \rho_0 \mathbf{V}_h \cdot \mathbf{B} d\Omega_{\mathbf{X}} + \int_{\Gamma_{\mathbf{X}}} \mathbf{V}_h \cdot \mathbf{H} d\Gamma_{\mathbf{X}} \quad (3.13)$$

$$- \sum_{\alpha=1}^m \int_{\Omega_{\mathbf{X}}} \frac{1}{4} \left(\frac{d}{dt} \mathbf{\Gamma}_h^\alpha \right) : \mathbb{V}^\alpha : \left(\frac{d}{dt} \mathbf{\Gamma}_h^\alpha \right) d\Omega_{\mathbf{X}}, \quad (3.14)$$

which completes the proof. \square

This numerical stability property guarantees that the proposed modeling and computational framework preserve critical structures of the physical system. It is worth emphasizing that the energy stable scheme is constructed based on the thermodynamically consistent continuum model derived in Section 2. To the best of our knowledge, we are unaware of any other numerical scheme with a priori energy stability proved for incompressible viscoelasticity. A straightforward consequence of this stability property is that the energy will be monotonically decreasing for unforced mechanical systems (i.e., $\mathbf{B} = \mathbf{0}$ and $\mathbf{H} = \mathbf{0}$).

For compressible materials, one may show that a pressure-squared term enters into the definition of the energy, providing boundedness of the pressure field. This justifies the use of equal-order interpolation for compressible materials. For fully incompressible materials, the pressure force has no contribution to the energy, and its stability comes from the inf-sup condition [65, Section 3.2]. The inf-sup stability of the discrete spaces given in this section has been numerically examined in [65, Section 4.1], where it was found that $\mathbf{b} + 1 \leq \mathbf{a}$ guarantees the element pair to be inf-sup stable. In this work, we use $\mathbf{a} = 1$ and $\mathbf{b} = 0$ for the discrete function spaces.

In addition to the energy stability, one may also conveniently show the momentum conservation of the semidiscrete formulation, which is stated in the following theorem.

Theorem 2 (Semidiscrete momentum conservation). *If $\Gamma_{\mathbf{x}}^t = \Gamma_{\mathbf{x}}^{h,t}$, the following conservation properties hold for the semidiscrete formulation (3.6)-(3.8),*

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \rho_0 \mathbf{V}_h d\Omega_{\mathbf{X}} & = \int_{\Omega_{\mathbf{X}}} \rho_0 \mathbf{B} d\Omega_{\mathbf{X}} + \int_{\Gamma_{\mathbf{X}}} \mathbf{H} d\Gamma_{\mathbf{X}}, \\
\frac{d}{dt} \int_{\Omega_{\mathbf{X}}} \rho_0 \boldsymbol{\varphi}_h \times \mathbf{V}_h d\Omega_{\mathbf{X}} & = \int_{\Omega_{\mathbf{X}}} \rho_0 \boldsymbol{\varphi}_h \times \mathbf{B} d\Omega_{\mathbf{X}} + \int_{\Gamma_{\mathbf{X}}} \boldsymbol{\varphi}_h \times \mathbf{H} d\Gamma_{\mathbf{X}}.
\end{aligned}$$

Proof. The above conservation properties are direct consequences of choosing $\mathbf{w}_h = \mathbf{e}_i$ and $\mathbf{w}_h = \mathbf{e}_i \times \boldsymbol{\varphi}_h$ respectively in (3.8), where \mathbf{e}_i is a unit vector in the i -th direction. \square

Remark 11. *It can be shown that the dissipation term in the stability estimate (3.12) can be equivalently written as*

$$\sum_{\alpha=1}^m \int_{\Omega_{\mathbf{X}}} \frac{1}{4} \left(\frac{d}{dt} \mathbf{\Gamma}_h^\alpha \right) : \mathbb{V}^\alpha : \left(\frac{d}{dt} \mathbf{\Gamma}_h^\alpha \right) d\Omega_{\mathbf{X}} = \sum_{\alpha=1}^m \int_{\Omega_{\mathbf{X}}} \mathbf{Q}^\alpha : (\mathbb{V}^\alpha)^{-1} : \mathbf{Q}^\alpha d\Omega_{\mathbf{X}}.$$

It should be pointed out that the stability is analyzed for the semi-discrete scheme. It remains an intriguing topic to further extend this estimate to the fully discrete regime. The energy-momentum scheme [104, 105, 106] is a promising candidate for this goal.

3.3. Time integration algorithm

In this section, we first introduce a discrete algorithm for updating the stresses. Following that, we state the fully discrete scheme using the generalized- α method.

3.3.1. Stress update algorithm

To obtain the stress, we need to perform time integration for the constitutive laws. Let the time interval $(0^+, T]$ be divided into N subintervals of size $\Delta t_n := t_{n+1} - t_n$ delimited by a discrete time vector $\{t_n\}_{n=0}^N$. The approximations to the velocity, pressure, and displacement and their first time derivatives at time t_n are denoted as

$$\mathbf{Y}_n := \{\mathbf{V}_n, P_n, \mathbf{U}_n\}^T \quad \text{and} \quad \dot{\mathbf{Y}}_n := \{\dot{\mathbf{V}}_n, \dot{P}_n, \dot{\mathbf{U}}_n\}^T,$$

respectively. Correspondingly, the approximations to the deformation gradient and strain measures at time t_n are represented as

$$\mathbf{F}_n = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}_n, \quad J_n = \det(\mathbf{F}_n), \quad \mathbf{C}_n = \mathbf{F}_n^T \mathbf{F}_n, \quad \tilde{\mathbf{C}}_n = J_n^{-2/3} \mathbf{C}_n.$$

The approximated projection tensor and elasticity tensor are given by

$$\mathbb{P}_{n+1} = \mathbb{I} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}, \quad \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} = 4J_{n+1}^{-\frac{4}{3}} \frac{\partial^2 G_{\text{iso}}^{\infty}(\tilde{\mathbf{C}}_{n+1})}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}}$$

The algorithmic stresses at time t_{n+1} read as

$$\begin{aligned} \mathbf{S}_{n+1} &= \mathbf{S}_{\text{iso } n+1} + \mathbf{S}_{\text{vol } n+1}, \quad \mathbf{S}_{\text{iso } n+1} = J_{n+1}^{-\frac{2}{3}} \mathbb{P}_{n+1} : \tilde{\mathbf{S}}_{n+1}, \quad \mathbf{S}_{\text{vol } n+1} = -J_{n+1} P_{n+1} \mathbf{C}_{n+1}^{-1}, \\ \tilde{\mathbf{S}}_{n+1} &= \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha}, \quad \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} = 2 \left(\frac{\partial G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}}} \right)_{n+1}, \quad \tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha} = \frac{J_{n+1}^{\frac{4}{3}} \beta_{\alpha}^{\infty}}{2\mu^{\alpha}} \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} : \mathbf{Q}_{n+1}^{\alpha}. \end{aligned}$$

To evaluate the stresses, we need to provide an algorithmic way to evaluate $\mathbf{Q}_{n+1}^{\alpha}$ based on the hereditary integral (2.44). Following the notation introduced in [47, 49], we first introduce a dimensionless parameter $\xi^{\alpha} := -\Delta t_n / 2\tau^{\alpha}$. The approximation to the variable \mathbf{Q}^{α} at time t_{n+1} is given by

$$\begin{aligned} \mathbf{Q}_{n+1}^{\alpha} &= \exp(-t_{n+1}/\tau^{\alpha}) \mathbf{Q}_0^{\alpha} + \int_{0^+}^{t_{n+1}} \exp(-(t_{n+1}-s)/\tau^{\alpha}) \beta_{\alpha}^{\infty} \frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^{\infty} ds \\ &= \exp(2\xi^{\alpha}) \exp(-t_n/\tau^{\alpha}) \mathbf{Q}_0^{\alpha} + \exp(2\xi^{\alpha}) \int_{0^+}^{t_n} \exp(-(t_n-s)/\tau^{\alpha}) \beta_{\alpha}^{\infty} \frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^{\infty} ds \\ &\quad + \int_{t_n}^{t_{n+1}} \exp(-(t_{n+1}-s)/\tau^{\alpha}) \beta_{\alpha}^{\infty} \frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^{\infty} ds \\ &= \exp(2\xi^{\alpha}) \mathbf{Q}_n^{\alpha} + \int_{t_n}^{t_{n+1}} \exp(-(t_{n+1}-s)/\tau^{\alpha}) \beta_{\alpha}^{\infty} \frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^{\infty} ds \\ &\approx \exp(2\xi^{\alpha}) \mathbf{Q}_n^{\alpha} + \exp(\xi^{\alpha}) \beta_{\alpha}^{\infty} \int_{t_n}^{t_{n+1}} \frac{d}{ds} \tilde{\mathbf{S}}_{\text{iso}}^{\infty} ds \\ &= \beta_{\alpha}^{\infty} \exp(\xi^{\alpha}) \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \exp(\xi^{\alpha}) \left(\exp(\xi^{\alpha}) \mathbf{Q}_n^{\alpha} - \beta_{\alpha}^{\infty} \tilde{\mathbf{S}}_{\text{iso } n}^{\infty} \right). \end{aligned} \tag{3.15}$$

In the second-to-last step of (3.15), a mid-point rule is applied for the exponential term in the time integral to obtain an approximation [47, 49]. Error analysis shows that the above approximation is second-order accurate, and the relation (3.15) is often referred to as the recurrence update formula for \mathbf{Q}^{α} . There exists an alternate second-order accurate recurrence update formula by applying the mid-point rule to the stress rate, rather than the exponential kernel term, in the time integral. Interested readers are referred to [47, p. 355] for details.

Remark 12. We mention that the recursive formula (3.15) is obtained by making use of the semigroup property of the kernel of the hereditary integral, and there exist other recursive formulas [47, Chapter 10]. A closely related recursive formula was originally developed by Herrmann and Peterson for the small deformation theory [107]. The recursive formula (3.15) is second-order accurate, and it is feasible to achieve higher-order accuracy with more involved update formulas for the internal state variables [99]. In fractional-order viscoelasticity models, however, a different recursive formula can be derived by invoking the fast convolution method [16].

Remark 13. In the analysis calculations, the values of \mathbf{Q}_{n+1}^α can be conveniently initialized and stored at quadrature points. In the constitutive routine, their values at the quadrature points are updated by (3.15), which are utilized as the input for the calculation of stresses at the quadrature points. Yet, in the postprocessing (such as visualization), the stresses are typically not sampled or interpolated at the quadrature points used for analysis. The global smoothing procedure [108, 109] can be invoked to recover the values of \mathbf{Q}_{n+1}^α and subsequently the values of stresses.

3.3.2. Fully discrete scheme

With the time discrete stress given above, we may state the fully discrete algorithm by invoking the generalized- α method [110, 111]. At time t_n , given \mathbf{Y}_n and $\dot{\mathbf{Y}}_n$, the time step size Δt_n , and the parameters α_m , α_f , and γ , find \mathbf{Y}_{n+1} and $\dot{\mathbf{Y}}_{n+1}$, such that for $\forall \{Q_h, \mathbf{W}_h\} \in \mathcal{V}_{P_h} \times \mathcal{V}_{V_h}$,

$$\mathbf{0} = \mathbf{B}^k \left(\dot{\mathbf{Y}}_{n+\alpha_m}, \mathbf{Y}_{n+\alpha_f} \right), \quad (3.16)$$

$$0 = \mathbf{B}^p \left(Q_h; \dot{\mathbf{Y}}_{n+\alpha_m}, \mathbf{Y}_{n+\alpha_f} \right), \quad (3.17)$$

$$0 = \mathbf{B}^m \left(\mathbf{W}_h; \dot{\mathbf{Y}}_{n+\alpha_m}, \mathbf{Y}_{n+\alpha_f} \right), \quad (3.18)$$

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \Delta t_n \dot{\mathbf{Y}}_n + \gamma \Delta t_n \left(\dot{\mathbf{Y}}_{n+1} - \dot{\mathbf{Y}}_n \right), \quad (3.19)$$

$$\dot{\mathbf{Y}}_{n+\alpha_m} = \dot{\mathbf{Y}}_n + \alpha_m \left(\dot{\mathbf{Y}}_{n+1} - \dot{\mathbf{Y}}_n \right), \quad (3.20)$$

$$\mathbf{Y}_{n+\alpha_f} = \mathbf{Y}_n + \alpha_f \left(\mathbf{Y}_{n+1} - \mathbf{Y}_n \right). \quad (3.21)$$

Let $\varrho_\infty \in [0, 1]$ denote the spectral radius of the amplification matrix at the highest mode. The following choice of the parameters ensures second-order accuracy, unconditional stability, and controllable high-frequency dissipation for linear first-order ordinary differential equations [111],

$$\alpha_m = \frac{1}{2} \left(\frac{3 - \varrho_\infty}{1 + \varrho_\infty} \right), \quad \alpha_f = \frac{1}{1 + \varrho_\infty}, \quad \gamma = \frac{1}{1 + \varrho_\infty}.$$

Remark 14. It is known that the many different temporal schemes may be recovered by the generalized- α scheme via distinct choices of the parameters. For example, choosing $\varrho_\infty = 0.0$ renders a scheme that is spectrally equivalent to the second-order backward difference method [111]; choosing $\varrho_\infty = 1.0$ recovers the mid-point rule. It is also worth pointing out that the generalized- α scheme has been conventionally applied to second-order structural dynamics. Recent work shows that the generalized- α method applied to a first-order structural dynamic system does not suffer from the ‘overshoot’ phenomenon [112], and thus possesses many desirable properties of implicit schemes, as noted by Hilber and Hughes [113]. Writing the structural dynamics problem as a first-order system introduces three additional velocity degrees of freedom per node. It can be shown that these additional degrees of freedom can be solved in a segregated manner in a consistent Newton-Raphson algorithm [60, 63, 65, 114]. The additional cost induced by the velocity degrees of freedom is merely the memory for storing them and an explicit update formula, which are thus marginal.

Remark 15. The stability of the generalized- α scheme was analyzed based on linear problems [110, 111] and remains unclear for nonlinear problems. In fact, it is known that the mid-point rule, an instantiation of the generalized- α scheme with $\varrho_\infty = 1.0$, is often energetically unstable for nonlinear structural dynamics

[115, 116]. It remains an interesting topic to construct fully discrete schemes that are provably stable in energy, and the family of energy-momentum methods serves as a promising candidate in this role [104, 115, 116, 117]

Remark 16. It is worth pointing out that the generalized- α scheme may achieve the claimed second-order accuracy if all unknowns are collocated at the intermediate time step, as is done in (3.16)-(3.21). In a very popular approach, the pressure is collocated at the time step t_{n+1} with the rest variables collocated at the intermediate time step following the rule of the generalized- α scheme. It was found recently that the claimed second-order accuracy is lost in that approach [69].

3.4. Elasticity tensor

Here we provide the isochoric elasticity tensors that are used in the consistent linearization of the model. We first define the elasticity tensors approximated at time t_{n+1} as

$$\mathbb{C}_{\text{iso } n+1} := \left(2 \frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}} \right)_{n+1}, \quad \mathbb{C}_{\text{iso } n+1}^{\infty} := \left(2 \frac{\partial \mathbf{S}_{\text{iso}}^{\infty}}{\partial \mathbf{C}} \right)_{n+1}, \quad \text{and} \quad \mathbb{C}_{\text{neq } n+1}^{\alpha} := \left(2 \frac{\partial \mathbf{S}_{\text{neq}}^{\alpha}}{\partial \mathbf{C}} \right)_{n+1}.$$

Based on the additive split structure of the stress, the isochoric elasticity tensor $\mathbb{C}_{\text{iso } n+1}$ can be expressed as

$$\mathbb{C}_{\text{iso } n+1} = \left(2 \frac{\partial \mathbf{S}_{\text{iso}}^{\infty}}{\partial \mathbf{C}} \right)_{n+1} + \sum_{\alpha=1}^m \left(2 \frac{\partial \mathbf{S}_{\text{neq}}^{\alpha}}{\partial \mathbf{C}} \right)_{n+1} = \mathbb{C}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \mathbb{C}_{\text{neq } n+1}^{\alpha}.$$

It can be shown that

$$\begin{aligned} \mathbb{C}_{\text{iso } n+1}^{\infty} &= \mathbb{P}_{n+1} : \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} : \mathbb{P}_{n+1}^T + \frac{2}{3} \text{Tr} \left(J_{n+1}^{-\frac{2}{3}} \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} \right) \tilde{\mathbb{P}}_{n+1} - \frac{2}{3} (\mathbf{C}_{n+1}^{-1} \otimes \mathbf{S}_{\text{iso } n+1}^{\infty} + \mathbf{S}_{\text{iso } n+1}^{\infty} \otimes \mathbf{C}_{n+1}^{-1}), \\ \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} &:= 4J_{n+1}^{-\frac{4}{3}} \left(\frac{\partial^2 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1}, \quad \text{Tr}(\cdot) = (\cdot) : \mathbf{C}_{n+1}, \quad \tilde{\mathbb{P}}_{n+1} := \mathbf{C}_{n+1}^{-1} \odot \mathbf{C}_{n+1}^{-1} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}^{-1}. \end{aligned}$$

The derivation of the formula can be found in [49, p. 255]. Following similar steps, we have

$$\mathbb{C}_{\text{neq } n+1}^{\alpha} = \mathbb{P}_{n+1} : \tilde{\mathbb{C}}_{\text{neq } n+1}^{\alpha} : \mathbb{P}_{n+1}^T + \frac{2}{3} \text{Tr} \left(J_{n+1}^{-\frac{2}{3}} \tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha} \right) \tilde{\mathbb{P}}_{n+1} - \frac{2}{3} (\mathbf{C}_{n+1}^{-1} \otimes \mathbf{S}_{\text{neq } n+1}^{\alpha} + \mathbf{S}_{\text{neq } n+1}^{\alpha} \otimes \mathbf{C}_{n+1}^{-1}),$$

where

$$\tilde{\mathbb{C}}_{\text{neq } n+1}^{\alpha} := 2J_{n+1}^{-\frac{4}{3}} \left(\frac{\partial \tilde{\mathbf{S}}_{\text{neq}}^{\alpha}}{\partial \tilde{\mathbf{C}}} \right)_{n+1}.$$

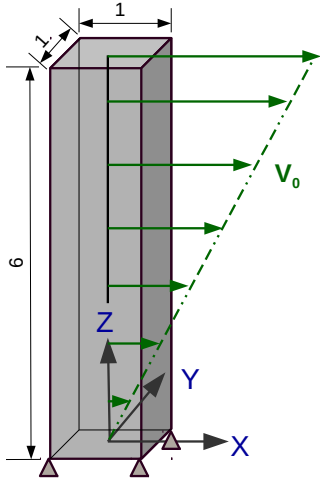
Based on (3.15), we have

$$2 \left(\frac{\partial \mathbf{Q}^{\alpha}}{\partial \mathbf{C}} \right)_{n+1} = \delta_{\alpha} J_{n+1}^{\frac{4}{3}} \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty},$$

with $\delta_{\alpha} := \beta_{\alpha}^{\infty} \exp(\xi^{\alpha})$. We may now express $\tilde{\mathbb{C}}_{\text{neq } n+1}^{\alpha}$ as

$$\begin{aligned} \tilde{\mathbb{C}}_{\text{neq } n+1}^{\alpha} &= \frac{4\beta_{\alpha}^{\infty} J_{n+1}^{-\frac{4}{3}}}{\mu^{\alpha}} \left(\frac{\partial^3 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1} : \mathbf{Q}_{n+1}^{\alpha} + \frac{2\delta_{\alpha} \beta_{\alpha}^{\infty}}{\mu^{\alpha}} \left(\frac{\partial^2 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1} : \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} \\ &= \frac{4\beta_{\alpha}^{\infty} J_{n+1}^{-\frac{4}{3}}}{\mu^{\alpha}} \left(\frac{\partial^3 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1} : \mathbf{Q}_{n+1}^{\alpha} + \frac{\delta_{\alpha} \beta_{\alpha}^{\infty} J_{n+1}^{\frac{4}{3}}}{2\mu^{\alpha}} \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} : \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty}. \end{aligned}$$

The algorithms for calculating the stresses and elasticity tensors for the IPC, HS, and MIPC models are documented in Appendix B, Appendix C, and Appendix D, respectively.



Material properties:

$$G_{\text{iso}}^{\infty} = \frac{c_1}{2} (\tilde{I}_1 - 3) + \frac{c_2}{2} (\tilde{I}_2 - 3),$$

$$\rho_0 = 1.1 \times 10^3 \text{ kg/m}^3,$$

$$E = 1.7 \times 10^7 \text{ Pa}, \quad c_1 = c_2 = E/6,$$

$$\beta_1^{\infty} = 1.0, \quad \mu^1 = 10c_1, \quad \tau^1 = 1 \text{ s}.$$

Reference scales:

$$L_0 = 1 \text{ m}, \quad M_0 = 1 \text{ kg}, \quad T_0 = 1 \text{ s}.$$

Table 1: Three-dimensional beam bending: problem setting, boundary conditions, initial conditions, and material properties. Notice that the parameter β_1^{∞} is only used in the IPC and MIPC models.

4. Numerical results

In this section, we investigate the proposed viscoelastic model by a suite of numerical examples using the numerical scheme proposed in the previous section. Unless otherwise specified, we use $\mathbf{p} + \mathbf{a} + 1$ Gauss quadrature points in each direction. Also recall that we have fixed $\mathbf{a} = 1$ and $\mathbf{b} = 0$ in the construction of discrete function spaces. We take $\mathbf{Q}_0^{\alpha} = \mathbf{O}$ in all numerical studies, which implies $\hat{\mathbf{S}}_0^{\alpha} = \hat{\mathbf{S}}_{\text{iso}0}^{\alpha}$.

4.1. Beam bending

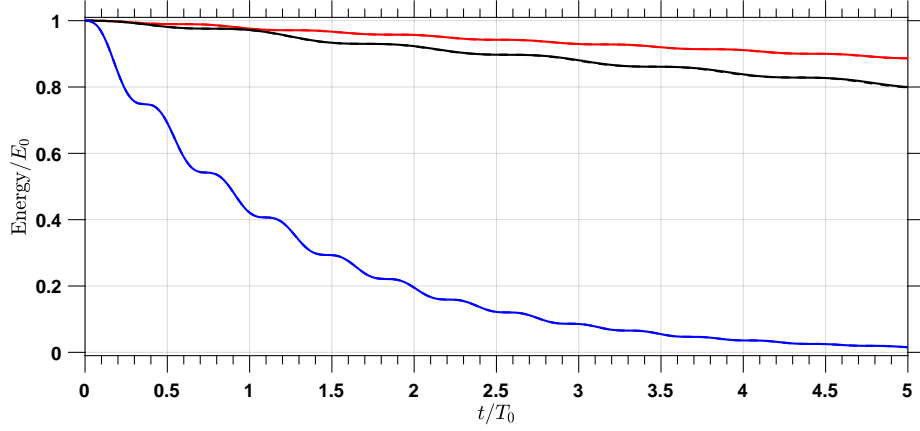
In this example, we consider a three-dimensional beam vibration problem with bending dominated deformation. The problem setting and the material properties are defined in Table 1. In particular, the ground elastic material is characterized by the Mooney-Rivlin model. The bottom surface of the beam is fully clamped while the rest boundary surfaces are specified with zero traction boundary conditions. The body is initially in a stress-free condition with zero displacements. The initial velocity is given by

$$\mathbf{V}(\mathbf{X}, 0) = \left(V_0 \frac{Z}{L_0}, 0, 0 \right)^T, \quad V_0 = \frac{5}{3} \text{ m/s},$$

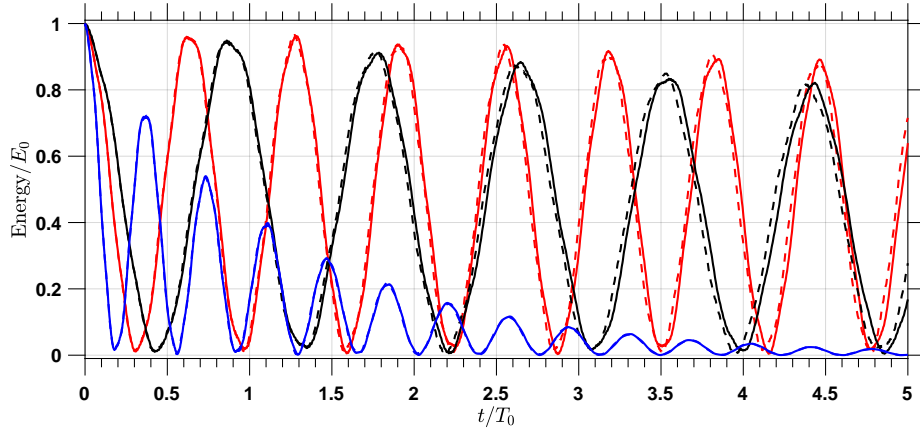
which initiates the vibration. For this specific problem, the energy stability given by Theorem 1 suggests that the total energy monotonically decreases with respect to time, i.e.,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_{\mathbf{x}}} \frac{1}{2} \rho_0 \|\mathbf{V}_h\|^2 + G_{\text{iso}}^{\infty}(\tilde{\mathbf{C}}_h) + \Upsilon^1(\tilde{\mathbf{C}}_h, \mathbf{\Gamma}_h^1) d\Omega_{\mathbf{x}} &= - \int_{\Omega_{\mathbf{x}}} \frac{1}{4} \left(\frac{d}{dt} \mathbf{\Gamma}_h^1 \right) : \mathbb{V}^1 : \left(\frac{d}{dt} \mathbf{\Gamma}_h^1 \right) d\Omega_{\mathbf{x}} \\ &= - \int_{\Omega_{\mathbf{x}}} \mathbf{Q}^1 : (\mathbb{V}^1)^{-1} : \mathbf{Q}^1 d\Omega_{\mathbf{x}} \leq 0. \end{aligned}$$

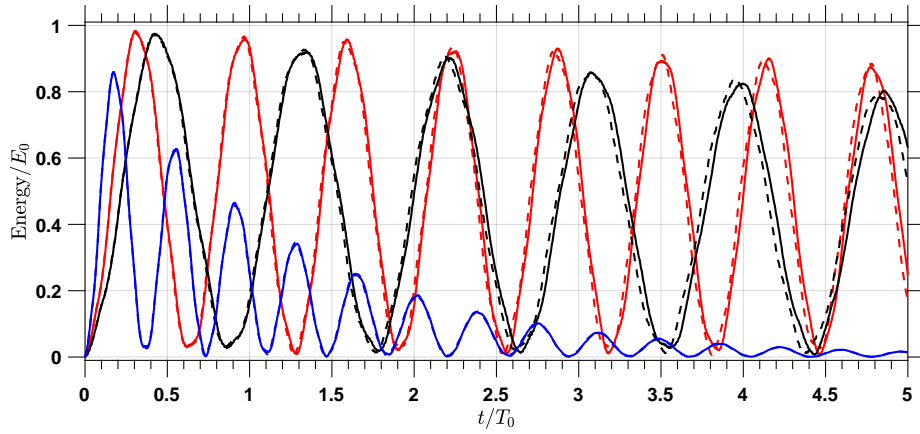
In the first set of simulations, we investigate the problem with two spatial meshes and integrate in time until $5T_0$. The generalized- α method is utilized with a fixed time step size and $\varrho_{\infty} = 1.0$, which recovers the mid-point rule. The evolution of the total, kinetic, and potential energies is illustrated in Figure 3. We can observe the energy decay of all three material models with different dissipation rates. In Figure 4, the snapshots of the pressure field on the deformed configuration at time $t/T_0 = 2.5$ are depicted. It is worth pointing out that the pressure field calculated based on the coarse mesh is very close to that of the finer mesh, suggesting the spline-based technology can accurately capture the stress with a relatively coarse mesh.



(a) Total energies



(b) Kinetic energies



(c) Potential energies

Figure 3: The total, kinetic, and potential energies (i.e., $G_{\text{iso}}^\infty(\tilde{\mathcal{C}}_h) + \Upsilon^1(\tilde{\mathcal{C}}_h, \Gamma_h^1)$) of the IPC (red), HS (blue), and MIPC (black) models over time. The simulations are performed with a fixed time step size $\Delta t/T_0 = 1 \times 10^{-3}$. The solid lines illustrate results obtained from a spatial mesh with $\mathbf{p} = 2$, $\mathbf{a} = 1$, $\mathbf{b} = 0$, and $5 \times 5 \times 30$ elements; the dashed lines illustrate results obtained from a spatial mesh with $\mathbf{p} = 1$, $\mathbf{a} = 1$, $\mathbf{b} = 0$, and $1 \times 1 \times 6$ elements. The reference value of the total energy E_0 is chosen to be the total energy at time $t = 0$, which is $1.1 \times 10^5 \text{ kg m}^2/\text{s}^2$.

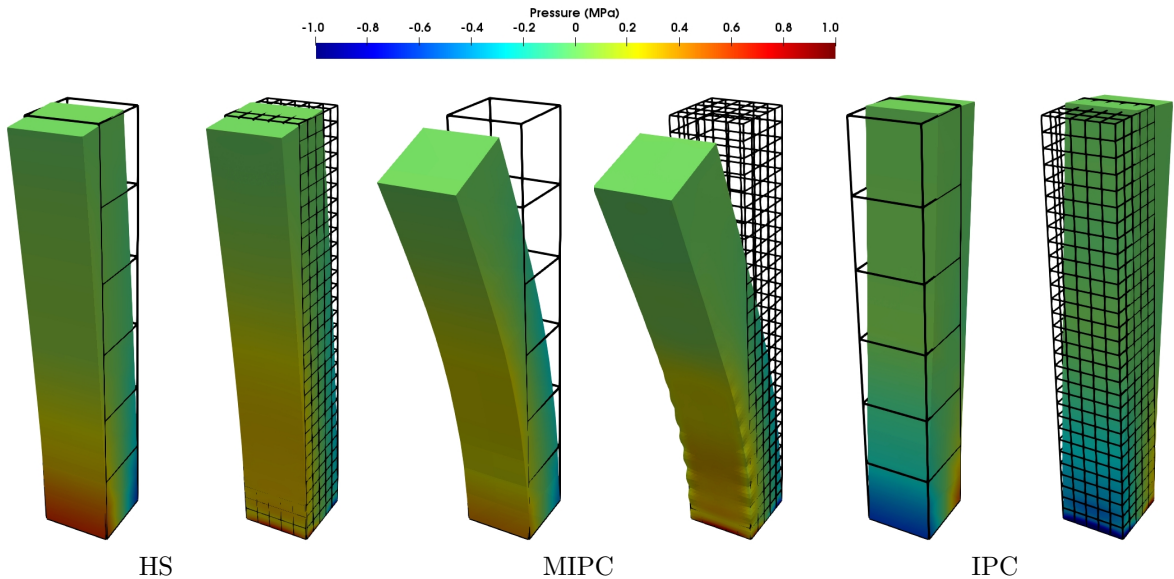


Figure 4: The snapshots of the pressure fields at time $t/T_0 = 2.5$ of the three models with a fixed time step size $\Delta t/T_0 = 1 \times 10^{-3}$ plotted on the deformed configuration. Two sets of spatial meshes are used: a spatial mesh with $\mathbf{p} = 2$, $\mathbf{a} = 1$, $\mathbf{b} = 0$, and $5 \times 5 \times 30$ elements and a spatial mesh with $\mathbf{p} = 1$, $\mathbf{a} = 1$, $\mathbf{b} = 0$, and $1 \times 1 \times 6$ elements. The meshes at the initial time are plotted as the black grid.

Yet, we also observe excessive oscillations for higher-order methods, as can be seen in the results of MIPC model shown in Figure 4. In Figure 5, the impact of the numerical dissipation from the generalized- α scheme is illustrated. The generalized- α schemes with $\Delta t/T_0 = 1.0 \times 10^{-3}$ and $\varrho_\infty = 0.0, 0.5$, and 1.0 are compared against the results of the mid-point rule with $\Delta t/T_0 = 1.0 \times 10^{-4}$. The numerical dissipation introduced by the time-stepping algorithm is negligible in comparison with the physical dissipation generated by the model from the energy evolution. In the detailed view shown in the three bottom figures of Figure 5, we see that, as expected, choosing a smaller value of ϱ_∞ induces more numerical dissipation since this parameter dictates the dissipation on the high-frequency modes according to the analysis of linear problems. The differences between the two simulations using $\varrho_\infty = 1.0$ with different time step sizes are indistinguishable.

We also report a suite of simulation results of the IPC and MIPC models with the modulus $\mu^1 = 0.2c_1$, with all other settings equal to those in Table 1. The snapshots of the deformation states and the pressure field are depicted in Figure 6. For the IPC model, the beam does not return to the original state as it reaches equilibrium. Instead, the beam is bent after $t/T_0 = 1.5$ and vibrates slightly around that bent configuration. The bent configuration is an unstable equilibrium configuration and eventually the simulation diverged. For the MIPC model, the beam eventually returns to the initial configuration with the energy dissipated due to the viscous effects. In Figure 7, the evolution of the energies is illustrated. The total energy of the IPC model reaches zero at $t/T_0 = 2.2$, after which the potential and total energies quickly become negative. Indeed, the IPC model cannot guarantee the boundedness of the configurational free energy (see, e.g. (2.50)). Thus, the evolution to unbounded negative energy is indeed possible for this very model, as is discovered in this case. This unstable behavior of the IPC model again suggests that the original IPC model, which results in non-vanishing non-equilibrium stresses, is materially unstable and may produce non-physical results.

4.2. Beam torsion

In this second example, we consider the torsion of a three-dimensional beam. The problem setting and material properties are illustrated in Table 2. In particular, the boundary conditions are identical to those of the beam bending example, and the body is initially stress-free with zero displacements. The initial velocity

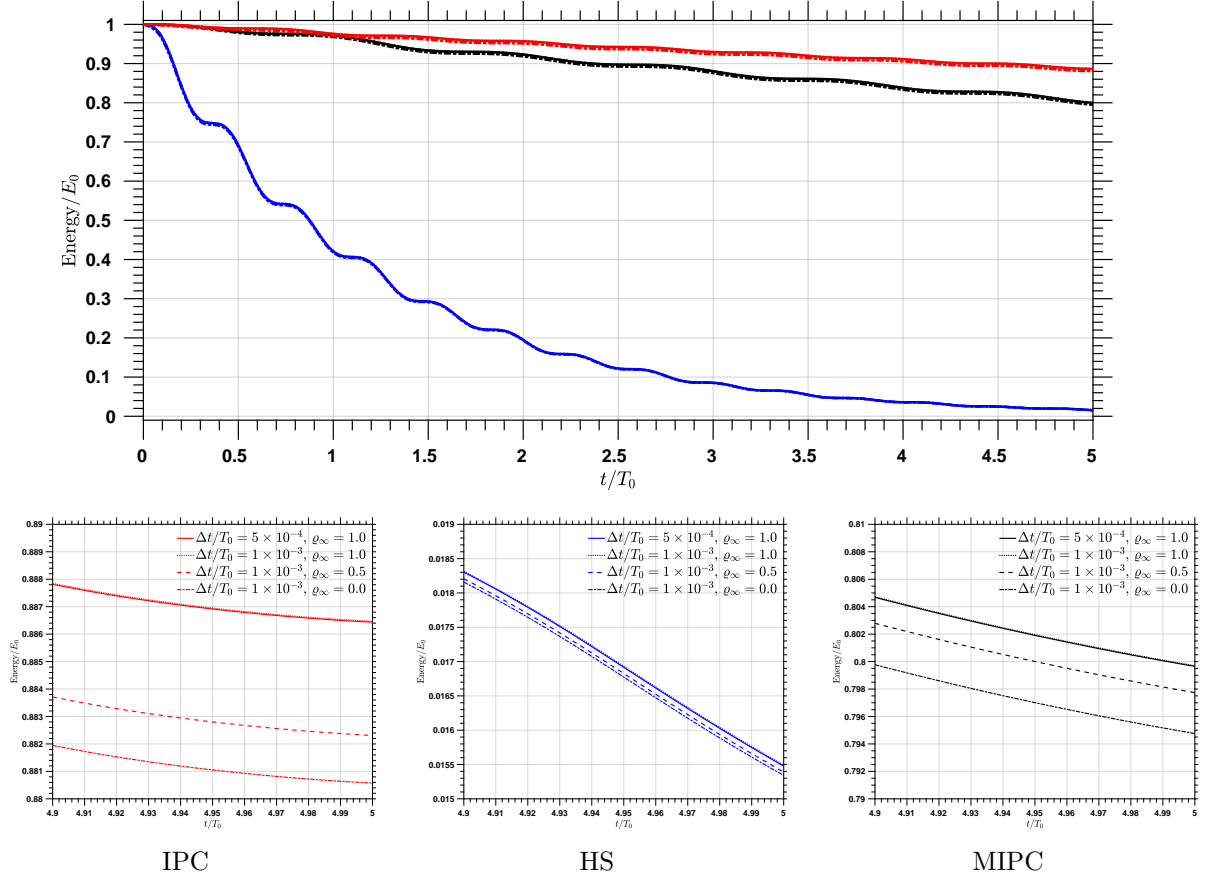


Figure 5: The total energies of the IPC (red), HS (blue), and MIPC (black) models over time. The simulations are performed with a spatial mesh with $p = 2$, $a = 1$, $b = 0$, and $5 \times 5 \times 30$ elements. The reference value E_0 is chosen to be the total energy at time $t/T_0 = 0$, which is $1.1 \times 10^5 \text{ kg m}^2/\text{s}^2$. Detailed view of the energies in the vicinity of $t/T_0 = 5$ is depicted in the bottom.

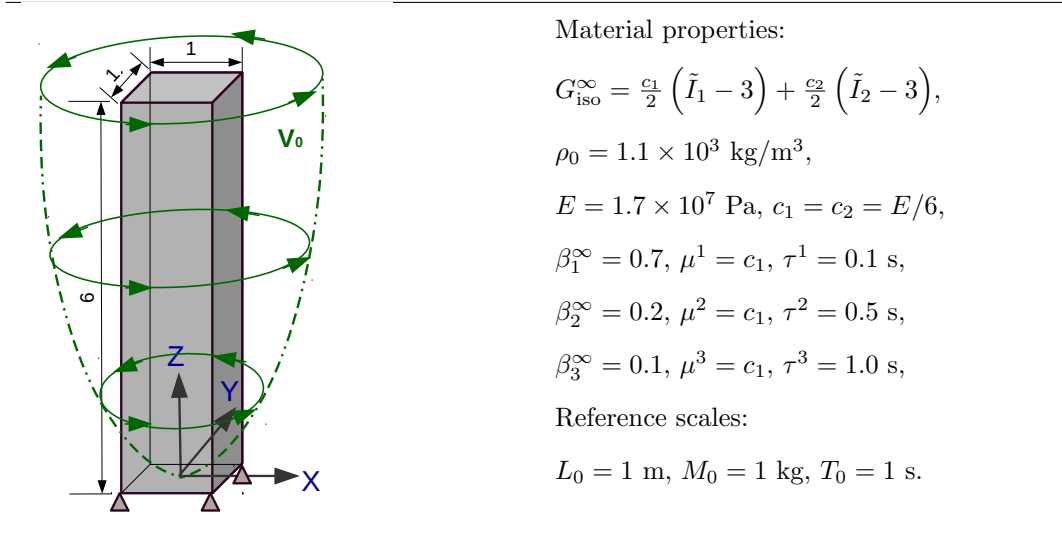


Table 2: Three-dimensional beam torsion: problem setting, boundary conditions, initial conditions, and material properties. Notice that the parameters β_1^{∞} , β_2^{∞} , and β_3^{∞} are only used in the IPC and MIPC models.

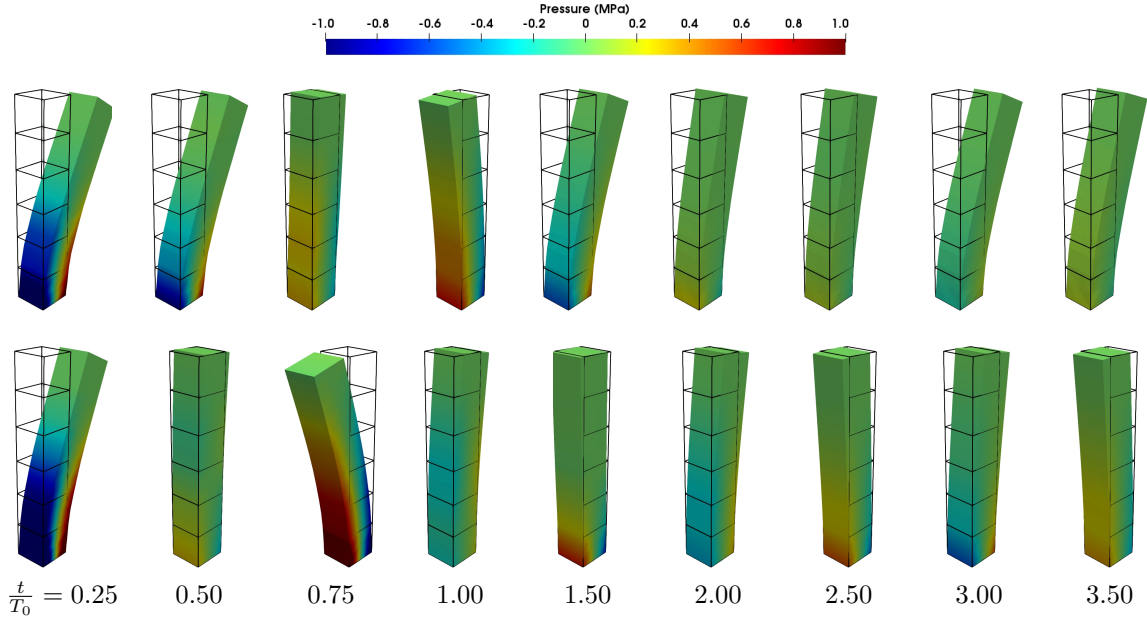


Figure 6: The snapshots of the pressure fields at nice time instances of the IPC (top) and MIPC (bottom) models with a fixed time step size $\Delta t/T_0 = 1 \times 10^{-3}$ plotted on the deformed configuration. The material parameters are the same as those reported in Table 1, except $\mu^1 = 0.2c_1$ here. The simulations are performed using a spatial mesh with $\mathbf{p} = 1$, $\mathbf{a} = 1$, $\mathbf{b} = 0$, and $1 \times 1 \times 6$ elements. The meshes at the initial time are plotted as the black grid.

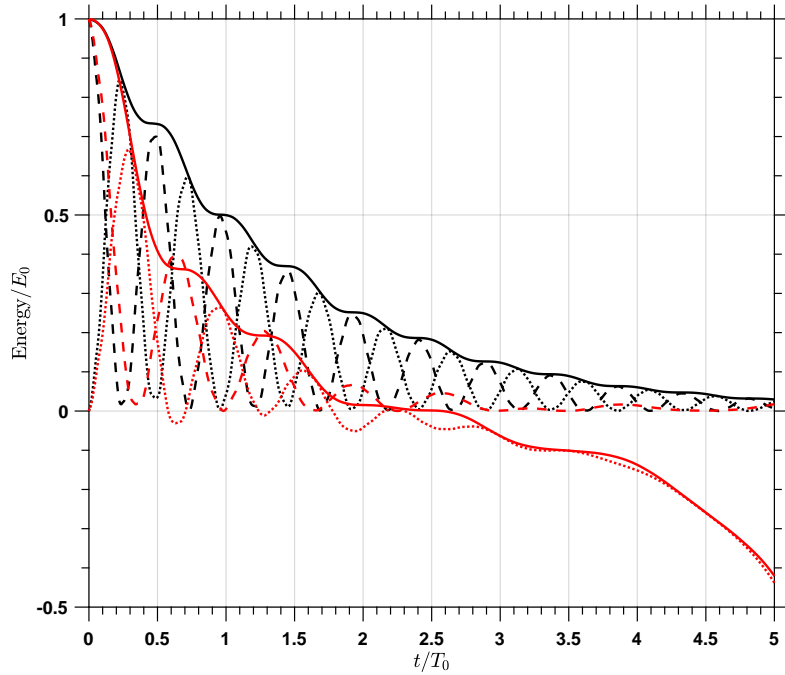
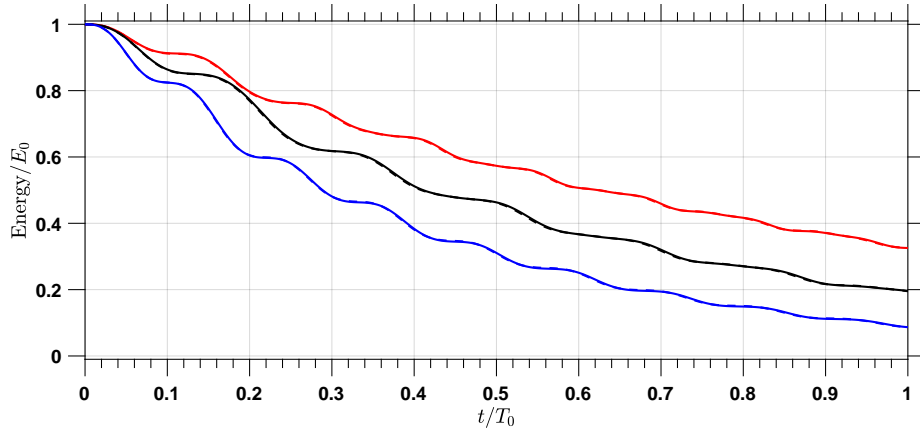
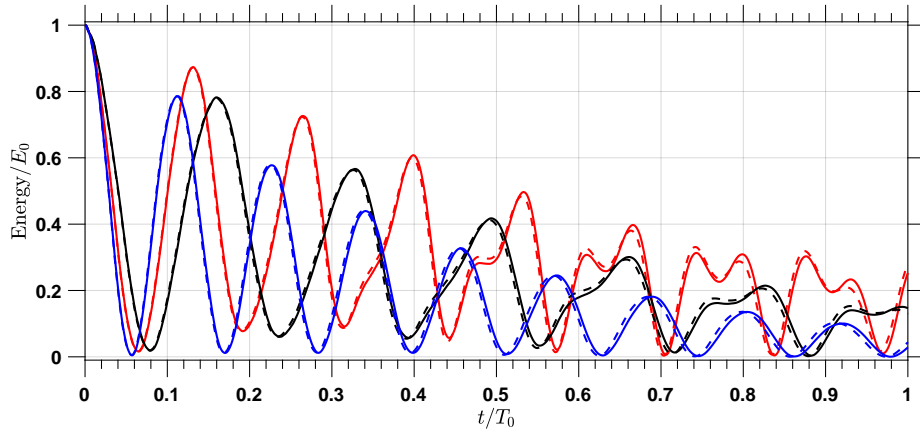


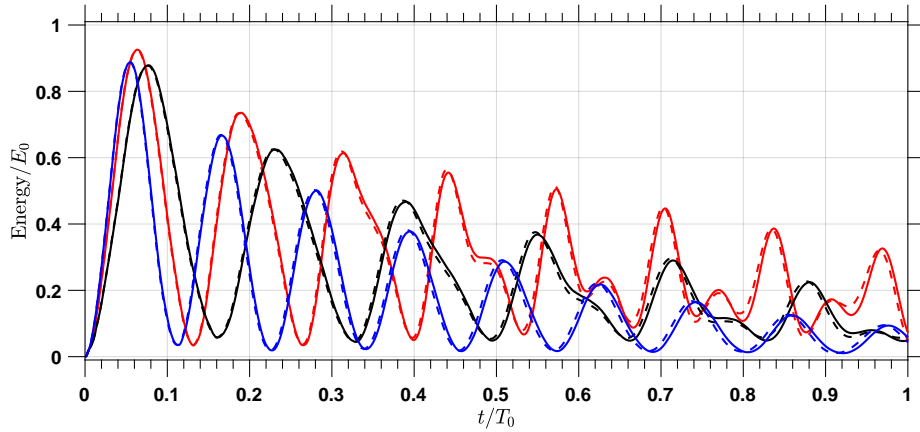
Figure 7: The total, kinetic, and potential energies of the IPC (red) and MIPC (black) models over time. The simulations are performed with $\mu^1 = 0.2c_1$, a fixed time step size $\Delta t/T_0 = 1 \times 10^{-3}$, and a spatial mesh with $\mathbf{p} = 1$, $\mathbf{a} = 1$, $\mathbf{b} = 0$, and $1 \times 1 \times 6$ elements. The solid, dashed, and dotted lines represent the total, kinetic, and potential energies, respectively. The reference value of the total energy E_0 is chosen to be the total energy at time $t = 0$, which is $1.1 \times 10^5 \text{ kg m}^2/\text{s}^2$.



(a) Total energies



(b) Kinetic energies



(c) Potential energies

Figure 8: The total, kinetic, and potential energies (i.e., $G_{\text{iso}}^\infty(\tilde{\mathbf{C}}_h) + \Upsilon^1(\tilde{\mathbf{C}}_h, \mathbf{\Gamma}_h^1)$) of the IPC (red), HS (blue), and MIPC (black) models over time for the beam torsion problem. The solid and dashed lines illustrate results obtained from the fine and coarse spatiotemporal discretizations. The reference value of the total energy E_0 is chosen to be the total energy at time $t = 0$, which is $2.75 \times 10^6 \text{ kg m}^2/\text{s}^2$.

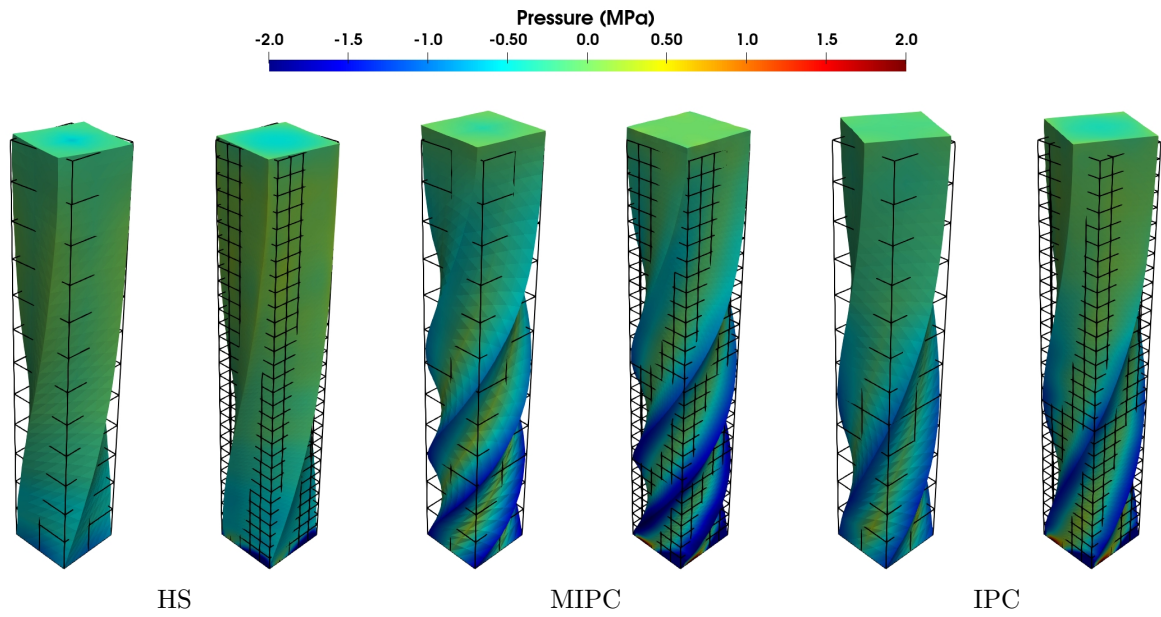


Figure 9: The snapshots of the pressure fields plotted on the deformed configuration at time $t/T_0 = 0.1$ of the three models for the beam torsion problem. The results from the coarse and fine discretizations are shown, with the meshes at the initial time plotted as the black grid.

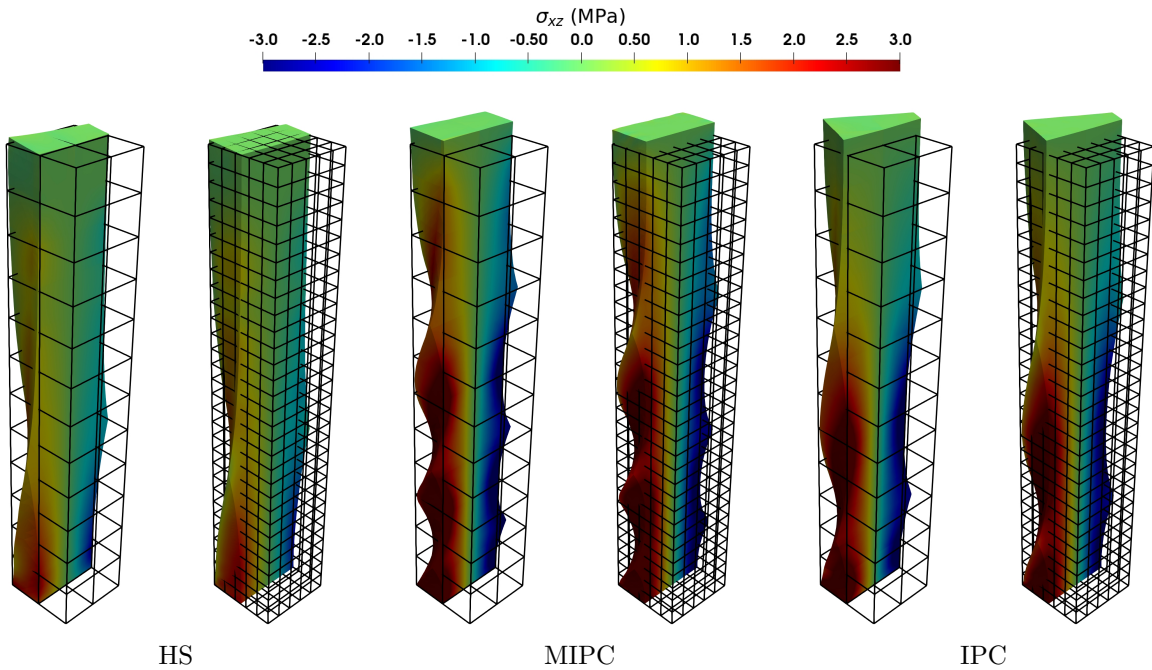


Figure 10: The cross-sectional view of the stress component σ_{xz} plotted on the deformed configuration at time $t/T_0 = 0.1$ of the three models for the beam torsion problem. The results from the coarse and fine discretizations are shown, with the meshes at the initial time plotted as the black grid.

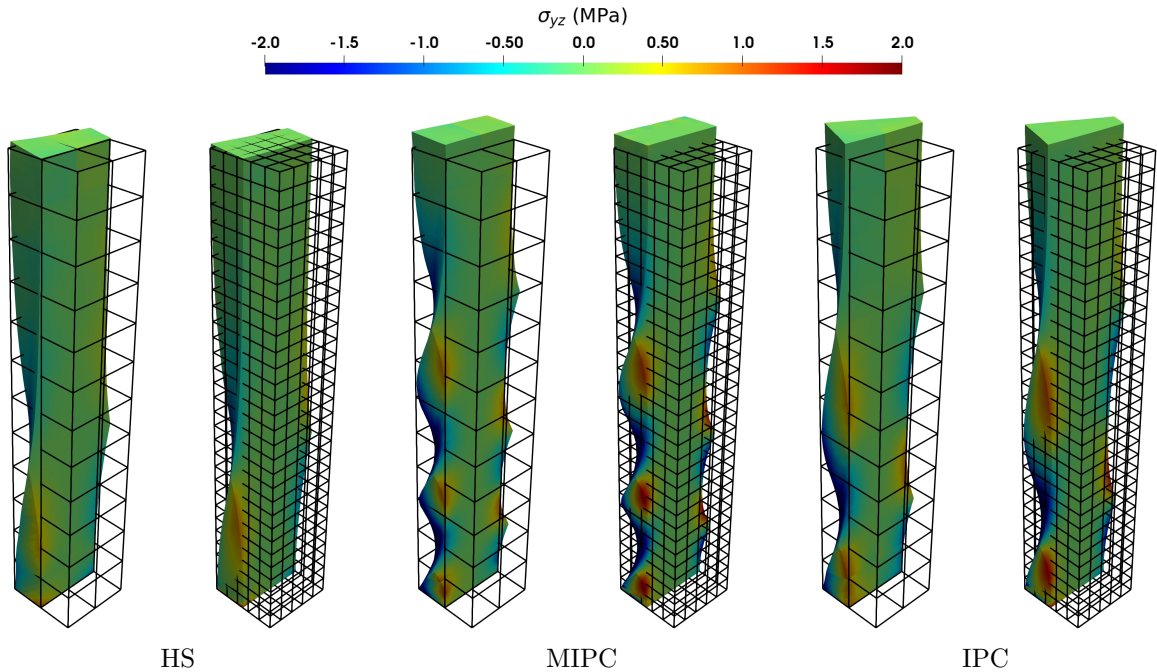


Figure 11: The cross-sectional view of the stress component σ_{yz} plotted on the deformed configuration at time $t/T_0 = 0.1$ of the three models for the beam torsion problem. The results from the coarse and fine discretizations are shown, with the meshes at the initial time plotted as the black grid.

is given by

$$\mathbf{V}(\mathbf{X}, 0) = V_0 \left(-\frac{Y}{L_0}, \frac{X}{L_0}, 0 \right)^T, \quad V_0 = 100 \sin \left(\frac{\pi Z}{12L_0} \right) \text{ m/s}.$$

Again, the problem constitutes an unforced mechanical system with the total energy monotonically decreasing with time. We numerically investigate this problem using two sets of spatiotemporal discretizations. In a coarse discretization, we use a fixed time step size $\Delta t/T_0 = 5.0 \times 10^{-4}$ and a spatial mesh with $\mathbf{p} = 1$ and $2 \times 2 \times 12$ elements; in a fine discretization, we use a fixed time step size $\Delta t/T_0 = 1.0 \times 10^{-4}$ and a spatial mesh with $\mathbf{p} = 2$ and $5 \times 5 \times 30$ elements. The time integration is performed with the mid-point rule (i.e., $\varrho_\infty = 1.0$). The evolutions of the total, kinetic, and potential energies are illustrated in Figure 8, from which we may observe that the coarse and fine meshes produce indistinguishable results in terms of the energies. Again the total energy is justified to be monotonically decreasing with time from the numerical results, which corroborates the estimate made in Theorem 1. In Figure 9, the snapshots of the pressure are plotted on the current configuration at time $t/T_0 = 0.1$. In Figures 10 and 11, the cross-sectional views of σ_{xz} and σ_{yz} are, respectively, depicted at time $t/T = 0.1$, where the cross-section is made at the $X = 0$ plane. For comparison purposes, the results of the three models calculated by the two discretizations are illustrated. We observe that both the pressure and the Cauchy stress are well resolved by the coarse mesh and are indistinguishable when comparing between the coarse- and fine-mesh results. This again demonstrates the superior capability of the chosen element technology in resolving the stresses.

4.3. Clamped cylindrical support

In the third example, we consider a clamped cylindrical support characterized by viscoelastic material behaviors with its inner surface connected to a vibrating device. This is a benchmark problem designed for examining the rate-dependent behavior with hysteresis loops [23, 118]. The geometrical setting of the

cylindrical support together with its material properties are illustrated in Table 3. The vibration of the device is described by

$$\mathbf{U}(\mathbf{X}, t) = (U_0 \sin(\omega t), 0, 0)^T, \quad U_0 = 5.625 \times 10^{-3} \text{m},$$

which represents a translation along the x -direction, and, in this study, the strain rate ω takes the values of 5 s^{-1} , 10 s^{-1} , and 20 s^{-1} , respectively. The outer surface of the support is fixed while traction-free boundary conditions are prescribed on the two end annular surfaces. The geometry of this cylindrical support can be exactly represented via NURBS. We use 24 elements in the circumferential direction, 6 elements in the radial direction, and 5 elements in the axial direction. We choose $\mathbf{p} = 2$ and 3 for the discrete pressure function space, respectively, to generate two sets of meshes. For the given strain rate, the period of one cycle is $2\pi/\omega$, and we simulate the problems for three cycles. For the mesh with $\mathbf{p} = 2$, the time step size is $\Delta t = 2.0 \times 10^{-4} \text{ s}$; for the mesh with $\mathbf{p} = 3$, the time step size is $\Delta t = 1.0 \times 10^{-4}$. In this problem, we choose $\varrho_\infty = 0.0$ to achieve better algorithm robustness.

	<p>Material properties:</p> $G_{\text{iso}}^\infty = \frac{c_1}{2} (\tilde{I}_1 - 3) + \frac{c_2}{2} (\tilde{I}_2 - 3),$ $\rho_0 = 1.1 \times 10^3 \text{ kg/m}^3,$ $E = 1.2 \times 10^5 \text{ Pa}, \quad c_1 = c_2 = E/6,$ $\beta_1^\infty = 1.0, \quad \mu^1 = 4c_1, \quad \tau^1 = 0.4 \text{ s},$ <p>Reference scales:</p> $L_0 = 1 \text{ m}, \quad M_0 = 1 \text{ kg}, \quad T_0 = 1 \text{ s}.$
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Table 3: Three-dimensional clamped cylindrical support: problem setting, boundary conditions, initial conditions, and material properties. Notice that the parameter β_1^∞ is only used in the MIPC models.

To investigate the viscous dissipative effect, we calculate the force on the inner surface Γ_{inner} as

$$\mathcal{F} := \int_{\Gamma_{\text{inner}}} \mathbf{P} \mathbf{N} d\Gamma.$$

The lateral component in the x -direction \mathcal{F}_x is plotted against the lateral displacement \mathbf{U}_x for the HS and MIPC models in Figure 12 for both meshes. The hysteretical curves are almost indistinguishable, suggesting the results are mesh independent. With the increase of the strain rate ω , both material models need more cycles to reach equilibrium; the increase in strain rate leads to an increase in force on the inner surface at the maximum strain; larger strain rates make the viscous effect less significant, making the material behavior closer to hyperelasticity that includes both equilibrium and non-equilibrium contributions. Those observations match well with prior studies [26, 35]. The pressure fields on the deformed configuration when the support reaches the maximum strain are depicted in Figure 13. The results from the discretization with $\mathbf{p} = 2$ and 3 show good agreement for both models under the three strain rates. This again demonstrates the superior stress resolving property of the proposed discretization method [63].

5. Conclusion

In this work, we start by considering a general continuum theory for viscoelasticity with the viscous deformation characterized by a set of internal state variables. With the incompressible constraint as a

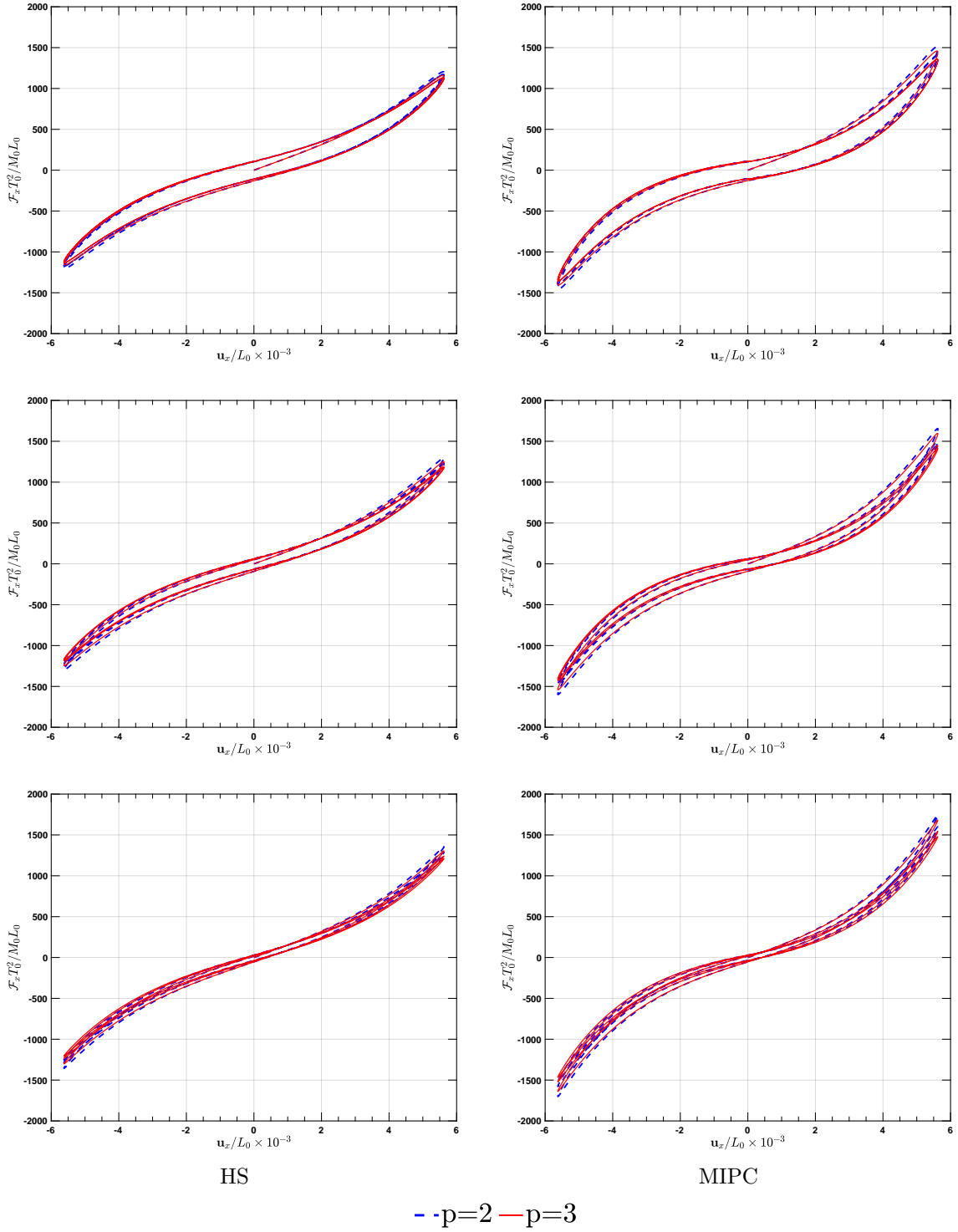


Figure 12: The lateral force \mathcal{F}_x on Γ_{inner} plotted against the lateral displacement \mathbf{U}_x for $\omega = 5\text{s}^{-1}$ (top row), $\omega = 10\text{s}^{-1}$ (middle row), $\omega = 20\text{s}^{-1}$ (bottom row). In the left column, the configurational free energy is given by the HS model while on the right column it is given by the MIPC model.

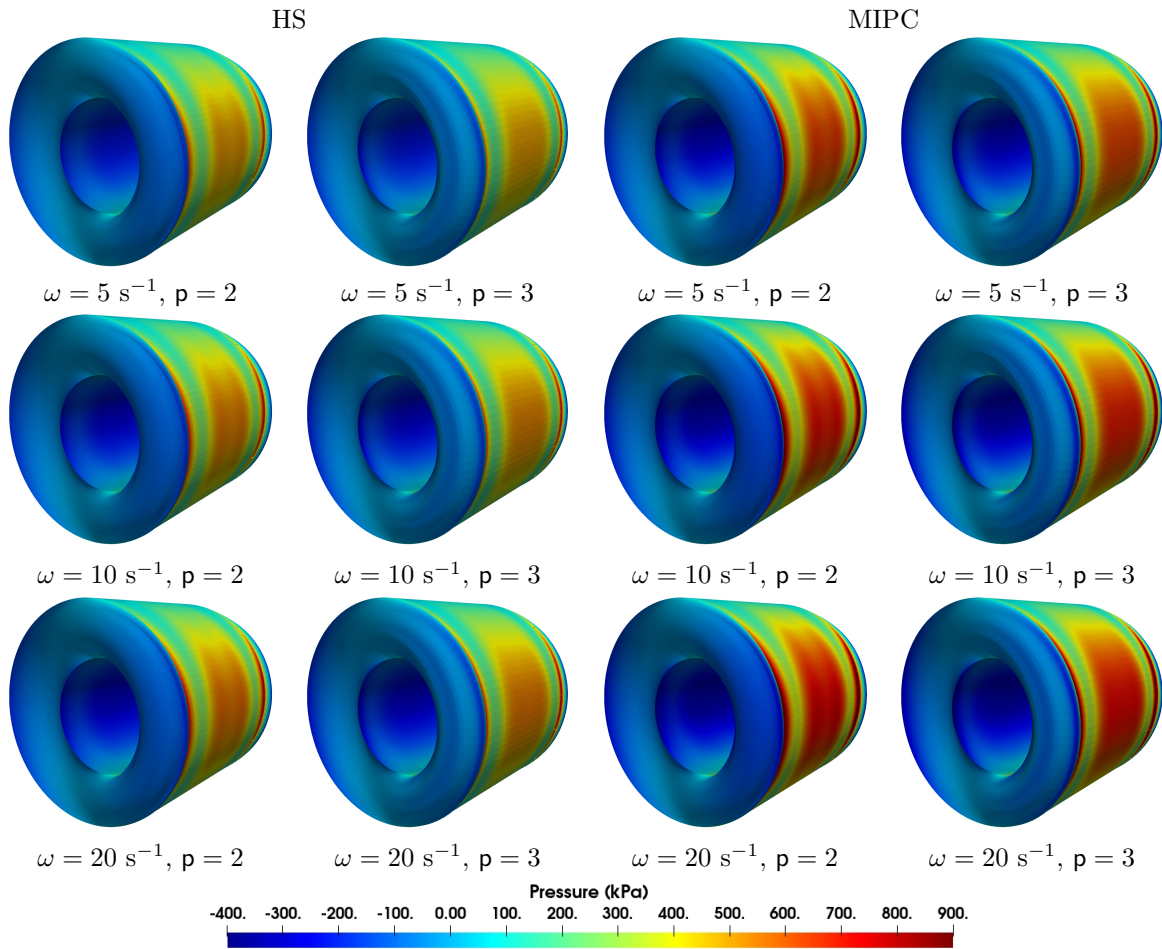


Figure 13: The snapshots of the pressure plotted on the deformed configuration at time $t = \pi/2\omega$ of the HS and MIPC models for the clamped cylindrical support problem. The results from the meshes with $p = 2$ and 3 are shown for comparison.

common material property in mind, we choose to develop a theory based on the Gibbs free energy [60, 65], which leads to a pressure primitive variable formulation. A set of nonlinear evolution equations is derived for the isochoric viscous deformation. With that, we consider a special form of the configurational free energy, which leads to the definition of the finite linear viscoelasticity. This viscoelasticity theory is endowed with a set of linear evolution equations, which is amenable to finite element implementation and anisotropic material modeling. It is revealed through the derivation that the original viscoelasticity model developed in [37, 47, 51] can be rectified to maintain thermodynamic consistency. In particular, the following rectifications need to be pointed out.

1. The right-hand side of the evolution equations (2.39) is driven by the *fictitious* second Piola-Kirchhoff stress $\tilde{\mathbf{S}}_{\text{iso}}^\alpha$. In contrast, in the original model, the right-hand side is driven by $J^{-\frac{2}{3}}\mathbb{P} : \tilde{\mathbf{S}}_{\text{iso}}^\alpha$ [37, 47].
2. The non-equilibrium stresses $\mathbf{S}_{\text{neq}}^\alpha$ and the conjugate variables \mathbf{Q}^α are two different quantities. They become identical only when the configurational free energy takes a special form (2.53).
3. The configurational free energy needs to satisfy the normalization condition (2.13); the relaxation of the non-equilibrium stresses poses an additional constraint on the form of the configurational free energy (see Section 2.4.3). In particular, the identical polymer chain model considered in [51] does not guarantee the vanishment of the viscous stress in the equilibrium limit, and we have identified an unstable solution of that model (see Figures 6 and 7).

Indeed, the original finite linear viscoelasticity model has been criticized for the lack of a thermomechanical foundation, and a finite time blow-up solution has been previously identified. The above three points could be the potential source of the blow-up phenomenon observed by S. Govindjee, et al. in [55]. Additionally, in one instantiation of the model, the free energy indicates that there is an *additive* split of the elastic and viscous Green-Lagrange strain (see (2.54)), resembling the elastoplasticity theory proposed by A. Green and P. Naghdi [57, 58]. Based on the consistent continuum theory, we construct a numerical formulation that inherits the stability to the semidiscrete formulation. We invoke the smooth generalization of the Taylor-Hood element based on NURBS for the spatial discretization. The well-known recurrence formula is utilized to integrate the constitutive relation, and the dynamic integration is performed by the generalized- α scheme. A variety of benchmark examples are presented and corroborate the properties of the continuum and numerical formulations in different deformation states. [The superior stress accuracy of the adopted element technology is demonstrated as well for different deformation modes.](#)

Based on this work, we will extend the proposed viscoelasticity theory to fiber-reinforced materials with a particular focus on arterial wall modeling and vascular fluid-structure interaction. It is also of interest to investigate the approximation of the viscoelastic constitutive relation directly from the hereditary integral, which may shed light on the numerical modeling of biological tissue growth and remodeling [19].

Acknowledgements

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Appendix A. An analysis of the null space of \mathbb{P}

In this section, we show that a fictitious stress living in the null space of \mathbb{P} has to be a zero stress. By definition, this stress satisfies

$$\mathbf{0} = \mathbb{P} : \tilde{\mathbf{S}} = \tilde{\mathbf{S}} - \frac{1}{3} \left(\tilde{\mathbf{S}} : \mathbf{C} \right) \mathbf{C}^{-1}.$$

From the above definition, one has

$$\tilde{\mathbf{S}} = \frac{1}{3} \left(\tilde{\mathbf{S}} : \mathbf{C} \right) \mathbf{C}^{-1} = \frac{1}{3} \left(\tilde{\mathbf{S}} : \tilde{\mathbf{C}} \right) \tilde{\mathbf{C}}^{-1} = a \tilde{\mathbf{C}}^{-1}, \quad \text{with } a := \frac{1}{3} \left(\tilde{\mathbf{S}} : \tilde{\mathbf{C}} \right). \quad (\text{A.1})$$

Recall that the fictitious stress is given by an isochoric energy $G_{\text{iso}}(\tilde{\mathbf{C}}, \Theta, \Gamma^1, \dots, \Gamma^m)$,

$$\tilde{\mathbf{S}} = 2 \frac{\partial G_{\text{iso}}}{\partial \tilde{\mathbf{C}}}.$$

Now, let $\mathbf{C}(t)$ be an arbitrary Green-Lagrange tensor that smoothly varies over time t . We consider a free energy G defined for $\mathbf{C}(t)$ with a conjugate stress satisfying the relation (A.1). Therefore, one has

$$\begin{aligned} G(\mathbf{C}(t)) - G(\mathbf{I}) &= \frac{1}{2} \int_0^t \mathbf{S}(\mathbf{C}(s)) : \dot{\mathbf{C}}(s) ds = \frac{1}{2} \int_0^t a \mathbf{C}^{-1}(s) : \dot{\mathbf{C}}(s) ds \\ &= \int_0^t a \mathbf{C}^{-1}(s) : \mathbf{F}^T \mathbf{d} \mathbf{F} ds = \int_0^t a \text{tr}[\mathbf{d}] ds, \end{aligned}$$

with \mathbf{d} being the rate of deformation tensor. Now we consider that the Green-Lagrange tensor characterizes an isochoric motion, implying $\text{tr}[\mathbf{d}] = 0$. Therefore, we have $G(\tilde{\mathbf{C}}) = 0$ for $\tilde{\mathbf{C}}$ characterizing volume-preserving deformations. This suggests that for a fictitious stress satisfying the relation (A.1), it has to be a zero stress,

$$\tilde{\mathbf{S}} = 2 \frac{\partial G_{\text{iso}}}{\partial \tilde{\mathbf{C}}} = \mathbf{O}.$$

In other words, an isochoric free energy cannot induce a hydrostatic fictitious stress.

Appendix B. Algorithm for the stress and elasticity tensor in the identical polymer chain model with $G^\alpha = F^\alpha$

In the original model based on the identical polymer chain assumption, it is taken that $G^\alpha(\tilde{\mathbf{C}}, \Theta) = F^\alpha(\tilde{\mathbf{C}}, \Theta) = \beta_\alpha^\infty G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta)$ [51]. Then the configurational free energy takes the following form.

$$\begin{aligned} \Upsilon^\alpha \left(\tilde{\mathbf{C}}, \Theta, \Gamma^1, \dots, \Gamma^m \right) &= H^\alpha(\Gamma^\alpha, \Theta) + \left(\hat{\mathbf{S}}_0^\alpha - 2 \frac{\partial G^\alpha(\tilde{\mathbf{C}}, \Theta)}{\partial \tilde{\mathbf{C}}} \right) : \frac{\Gamma^\alpha - \mathbf{I}}{2} + G^\alpha(\tilde{\mathbf{C}}, \Theta), \\ &= \mu^\alpha(\Theta) \left| \frac{\Gamma^\alpha - \mathbf{I}}{2} \right|^2 + \left(\hat{\mathbf{S}}_0^\alpha - \beta_\alpha^\infty \tilde{\mathbf{S}}_{\text{iso}}^\infty \right) : \frac{\Gamma^\alpha - \mathbf{I}}{2} + \beta_\alpha^\infty G_{\text{iso}}^\infty(\tilde{\mathbf{C}}, \Theta). \end{aligned}$$

To the best of our knowledge, numerical analysis for that model has not been performed, probably due to the appearance of a six-order tensor in the definition of the elasticity tensor. Here, we provide the algorithm for computing the stress as well as the elasticity tensor of this model. Given the time step Δt_n , and the value of $\tilde{\mathbf{S}}_{\text{iso } n}^\infty$ and \mathbf{Q}_n^α at each quadrature points, proceed through the following steps to compute the isochoric part of the second Piola-Kirchhoff stress and the isochoric part of the elasticity tensor.

1. Calculate the deformation gradient and the strain measures based on the displacement \mathbf{U}_{n+1} ,

$$\mathbf{F}_{n+1} = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}_{n+1}, \quad J_{n+1} = \det(\mathbf{F}_{n+1}), \quad \mathbf{C}_{n+1} = \mathbf{F}_{n+1}^T \mathbf{F}_{n+1}, \quad \tilde{\mathbf{C}}_{n+1} = J_{n+1}^{-2/3} \mathbf{C}_{n+1}.$$

2. Calculate the fictitious second Piola-Kirchhoff stress

$$\tilde{\mathbf{S}}_{\text{iso } n+1}^\infty = 2 \left(\frac{\partial G_{\text{iso}}^\infty}{\partial \tilde{\mathbf{C}}} \right)_{n+1}.$$

3. Calculate the fictitious elasticity tensor

$$\tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} = 4J_{n+1}^{-\frac{4}{3}} \left(\frac{\partial^2 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1}.$$

4. For $\alpha = 1, \dots, m$, calculate

$$\mathbf{Q}_{n+1}^{\alpha} = \beta_{\alpha}^{\infty} \exp(\xi^{\alpha}) \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \exp(\xi^{\alpha}) \left(\exp(\xi^{\alpha}) \mathbf{Q}_n^{\alpha} - \beta_{\alpha}^{\infty} \tilde{\mathbf{S}}_{\text{iso } n}^{\infty} \right).$$

5. For $\alpha = 1, \dots, m$, calculate

$$\tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha} = \beta_{\alpha}^{\infty} \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} - \frac{\beta_{\alpha}^{\infty}}{2\mu^{\alpha}} J_{n+1}^{\frac{4}{3}} \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} : \left(\beta_{\alpha}^{\infty} \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} - \hat{\mathbf{S}}_0^{\alpha} - \mathbf{Q}_{n+1}^{\alpha} \right).$$

6. Calculate the fictitious stress

$$\tilde{\mathbf{S}}_{n+1} = \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha}.$$

7. Calculate the projection tensor

$$\mathbb{P}_{n+1} = \mathbb{I} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}.$$

8. Calculate the isochoric part of the second Piola-Kirchhoff stress

$$\mathbf{S}_{\text{iso } n+1} = J_{n+1}^{-\frac{2}{3}} \mathbb{P}_{n+1} : \tilde{\mathbf{S}}_{n+1}.$$

9. Calculate the fictitious elasticity tensors for the non-equilibrium part,

$$\begin{aligned} \tilde{\mathbf{C}}_{\text{neq } n+1}^{\alpha} &= \beta_{\alpha}^{\infty} \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} - \frac{4\beta_{\alpha}^{\infty}}{\mu^{\alpha}} J_{n+1}^{-\frac{4}{3}} \left(\frac{\partial^3 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1} : \left(\beta_{\alpha}^{\infty} \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} - \hat{\mathbf{S}}_0^{\alpha} - \mathbf{Q}_{n+1}^{\alpha} \right) \\ &\quad - \frac{\beta_{\alpha}^{\infty}}{2\mu^{\alpha}} J_{n+1}^{\frac{4}{3}} (1 - \delta_{\alpha}) \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} : \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty}. \end{aligned}$$

10. Calculate the fictitious elasticity tensor

$$\tilde{\mathbf{C}}_{n+1} = \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbf{C}}_{\text{neq } n+1}^{\alpha}.$$

11. Calculate the fourth-order tensor

$$\tilde{\mathbb{P}}_{n+1} = \mathbf{C}_{n+1}^{-1} \odot \mathbf{C}_{n+1}^{-1} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}^{-1}.$$

12. Calculate the isochoric part of the elasticity tensor

$$\mathbf{C}_{\text{iso } n+1} = \mathbb{P}_{n+1} : \tilde{\mathbf{C}}_{n+1} : \mathbb{P}_{n+1}^T + \frac{2}{3} \text{Tr} \left(J_{n+1}^{-\frac{2}{3}} \tilde{\mathbf{S}}_{n+1} \right) \tilde{\mathbb{P}}_{n+1} - \frac{2}{3} (\mathbf{C}_{n+1}^{-1} \otimes \mathbf{S}_{\text{iso } n+1} + \mathbf{S}_{\text{iso } n+1} \otimes \mathbf{C}_{n+1}^{-1}).$$

Appendix C. Algorithm for the stress and elasticity tensor in the Holzapfel-Simo model

The Holzapfel-Simo model presented in Section 2.4.4 is different from the classical one documented in classical works [49, Chapter 6.10]. Here, we provide the algorithm for computing the stresses as well as the elasticity tensor. Given the time step Δt_n , and the value of $\tilde{\mathbf{S}}_{\text{iso } n}^{\infty}$ and \mathbf{Q}_n^{α} at each quadrature points, proceed through the following steps to compute the isochoric part of the second Piola-Kirchhoff stress and the isochoric part of the elasticity tensor.

1. Calculate the deformation gradient and the strain measures based on the displacement U_{n+1} ,

$$\mathbf{F}_{n+1} = \mathbf{I} + \nabla_{\mathbf{X}} U_{n+1}, \quad J_{n+1} = \det(\mathbf{F}_{n+1}), \quad \mathbf{C}_{n+1} = \mathbf{F}_{n+1}^T \mathbf{F}_{n+1}, \quad \tilde{\mathbf{C}}_{n+1} = J_{n+1}^{-2/3} \mathbf{C}_{n+1}.$$

2. Calculate

$$\tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} = 2 \left(\frac{\partial G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}}} \right)_{n+1}.$$

3. For $\alpha = 1, \dots, m$, calculate

$$\mathbf{Q}_{n+1}^{\alpha} = \mu^{\alpha} \exp(\xi^{\alpha}) \tilde{\mathbf{C}}_{n+1} + \exp(\xi^{\alpha}) \left(\exp(\xi^{\alpha}) \mathbf{Q}_n^{\alpha} - \mu^{\alpha} \tilde{\mathbf{C}}_n \right).$$

4. Calculate the fictitious stress

$$\tilde{\mathbf{S}}_{n+1} = \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \mathbf{Q}_{n+1}^{\alpha}.$$

5. Calculate the projection tensor

$$\mathbb{P}_{n+1} = \mathbb{I} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}.$$

6. Calculate the isochoric part of the second Piola-Kirchhoff stress

$$\mathbf{S}_{\text{iso } n+1} = J_{n+1}^{-\frac{2}{3}} \mathbb{P}_{n+1} : \tilde{\mathbf{S}}_{n+1}.$$

7. Calculate the fictitious elasticity tensors for the equilibrium and non-equilibrium parts,

$$\tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} = 4J_{n+1}^{-\frac{4}{3}} \left(\frac{\partial^2 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1}, \quad \text{and} \quad \tilde{\mathbf{C}}_{\text{neq } n+1}^{\alpha} = 2\mu^{\alpha} \exp(\xi^{\alpha}) J_{n+1}^{-\frac{4}{3}} \mathbb{I}, \quad \text{for } \alpha = 1, \dots, m.$$

8. Calculate the fictitious elasticity tensor

$$\tilde{\mathbf{C}}_{n+1} = \tilde{\mathbf{C}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbf{C}}_{\text{neq } n+1}^{\alpha}.$$

9. Calculate the fourth-order tensor

$$\tilde{\mathbb{P}}_{n+1} = \mathbf{C}_{n+1}^{-1} \odot \mathbf{C}_{n+1}^{-1} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}^{-1}.$$

10. Calculate the isochoric part of the elasticity tensor

$$\mathbf{C}_{\text{iso } n+1} = \mathbb{P}_{n+1} : \tilde{\mathbf{C}}_{n+1} : \mathbb{P}_{n+1}^T + \frac{2}{3} \text{Tr} \left(J_{n+1}^{-\frac{2}{3}} \tilde{\mathbf{S}}_{n+1} \right) \tilde{\mathbb{P}}_{n+1} - \frac{2}{3} (\mathbf{C}_{n+1}^{-1} \otimes \mathbf{S}_{\text{iso } n+1} + \mathbf{S}_{\text{iso } n+1} \otimes \mathbf{C}_{n+1}^{-1}).$$

Appendix D. Algorithm for the stress and elasticity tensor in the modified identical polymer chain model

The modified identical polymer chain model presented in Section 2.4.5 rectifies the non-physical behavior of the original identical polymer chain model in the thermodynamic equilibrium. Here we provide the algorithm for computing the stresses as well as the elasticity tensor of this model. Given the time step Δt_n , and the value of $\tilde{\mathbf{S}}_{\text{iso } n}^{\infty}$ and \mathbf{Q}_n^{α} at each quadrature points, proceed through the following steps to compute the isochoric part of the second Piola-Kirchhoff stress and the isochoric part of the elasticity tensor.

1. Calculate the deformation gradient and the strain measures based on the displacement U_{n+1} ,

$$\mathbf{F}_{n+1} = \mathbf{I} + \nabla_{\mathbf{X}} U_{n+1}, \quad J_{n+1} = \det(\mathbf{F}_{n+1}), \quad \mathbf{C}_{n+1} = \mathbf{F}_{n+1}^T \mathbf{F}_{n+1}, \quad \tilde{\mathbf{C}}_{n+1} = J_{n+1}^{-2/3} \mathbf{C}_{n+1}.$$

2. Calculate the fictitious second Piola-Kirchhoff stress

$$\tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} = 2 \left(\frac{\partial G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}}} \right)_{n+1}.$$

3. Calculate the fictitious elasticity tensor

$$\tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} = 4 J_{n+1}^{-4/3} \left(\frac{\partial^2 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1}.$$

4. For $\alpha = 1, \dots, m$, calculate

$$\mathbf{Q}_{n+1}^{\alpha} = \beta_{\alpha}^{\infty} \exp(\xi^{\alpha}) \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \exp(\xi^{\alpha}) \left(\exp(\xi^{\alpha}) \mathbf{Q}_n^{\alpha} - \beta_{\alpha}^{\infty} \tilde{\mathbf{S}}_{\text{iso } n}^{\infty} \right).$$

5. For $\alpha = 1, \dots, m$, calculate

$$\tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha} = \frac{J_{n+1}^{4/3} \beta_{\alpha}^{\infty}}{2\mu^{\alpha}} \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} : \mathbf{Q}_{n+1}^{\alpha}.$$

6. Calculate the fictitious stress

$$\tilde{\mathbf{S}}_{n+1} = \tilde{\mathbf{S}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbf{S}}_{\text{neq } n+1}^{\alpha}.$$

7. Calculate the projection tensor

$$\tilde{\mathbb{P}}_{n+1} = \mathbb{I} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}.$$

8. Calculate the isochoric part of the second Piola-Kirchhoff stress

$$\mathbf{S}_{\text{iso } n+1} = J_{n+1}^{-2/3} \tilde{\mathbb{P}}_{n+1} : \tilde{\mathbf{S}}_{n+1}.$$

9. Calculate the fictitious elasticity tensors for the non-equilibrium part,

$$\tilde{\mathbb{C}}_{\text{neq } n+1}^{\alpha} = \frac{4\beta_{\alpha}^{\infty} J_{n+1}^{-4/3}}{\mu^{\alpha}} \left(\frac{\partial^3 G_{\text{iso}}^{\infty}}{\partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}} \partial \tilde{\mathbf{C}}} \right)_{n+1} : \mathbf{Q}_{n+1}^{\alpha} + \frac{\delta_{\alpha} \beta_{\alpha}^{\infty} J_{n+1}^{4/3}}{2\mu^{\alpha}} \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} : \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty}.$$

10. Calculate the fictitious elasticity tensor

$$\tilde{\mathbb{C}}_{n+1} = \tilde{\mathbb{C}}_{\text{iso } n+1}^{\infty} + \sum_{\alpha=1}^m \tilde{\mathbb{C}}_{\text{neq } n+1}^{\alpha}.$$

11. Calculate the fourth-order tensor

$$\tilde{\mathbb{P}}_{n+1} = \mathbf{C}_{n+1}^{-1} \odot \mathbf{C}_{n+1}^{-1} - \frac{1}{3} \mathbf{C}_{n+1}^{-1} \otimes \mathbf{C}_{n+1}^{-1}.$$

12. Calculate the isochoric part of the elasticity tensor

$$\mathbb{C}_{\text{iso } n+1} = \tilde{\mathbb{P}}_{n+1} : \tilde{\mathbb{C}}_{n+1} : \tilde{\mathbb{P}}_{n+1}^T + \frac{2}{3} \text{Tr} \left(J_{n+1}^{-2/3} \tilde{\mathbf{S}}_{n+1} \right) \tilde{\mathbb{P}}_{n+1} - \frac{2}{3} (\mathbf{C}_{n+1}^{-1} \otimes \mathbf{S}_{\text{iso } n+1} + \mathbf{S}_{\text{iso } n+1} \otimes \mathbf{C}_{n+1}^{-1}).$$

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