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R_{cl}-spaces and closedness/completeness of certain function spaces in the topology of uniform convergence

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Abstract

It is shown that the notion of an R_{cl}-space (Demonstratio Math. 46(1) (2013), 229-244) fits well as a separation axiom between zero dimensionality and R_0-spaces. Basic properties of R_{cl}-spaces are studied and their place in the hierarchy of separation axioms that already exist in the literature is elaborated. The category of R_{cl}-spaces and continuous maps constitutes a full isomorphism closed, monoreflective (epireflective) subcategory of TOP. The function space $R_{cl}(X, Y)$ of all R_{cl}-supercontinuous functions from a space X into a uniform space Y is shown to be closed in the topology of uniform convergence. This strengthens and extends certain results in the literature (Demonstratio Math. 45(4) (2012), 947-952).

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1. Introduction

The notion of an R_{cl}-space evolved naturally in the study of R_{cl}-supercontinuous functions [37]. Here we study their basic properties and show that it fits well as a separation axiom between zero dimensionality and R_0-spaces. We reflect

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upon interrelations and interconnections that exist among R_{cl}-spaces and separation axioms which already exist in the lore of mathematical literature and lie between zero dimensionality and R_0-spaces. The class of R_{cl}-spaces properly contains each of the classes of zero dimensional spaces and ultra Hausdorff spaces [35] and is strictly contained in the class of R_0-spaces ([20, 33]) which in its turn properly contains each of the classes of functionally regular spaces ([3, 39]) and functionally Hausdorff spaces.

The organization of the paper is as follows: Section 2 is devoted to preliminaries and basic definitions. In Section 3 we elaborate upon the place of R_{cl}-spaces in the hierarchy of separation axioms which lie between zero dimensionality and R_0-spaces and already exist in the mathematical literature. Section 4 is devoted to study basic properties of R_{cl}-spaces wherein it is shown that (i) the property of being an R_{cl}-spaces is invariant under disjoint topological sums and initial sources so it is hereditary, productive, supinvariant, preimage invariant and projective; (ii) the category of R_{cl}-spaces and continuous maps is a full, isomorphism closed monoreflective (epireflective) subcategory of TOP; (iii) it is shown that a T_0-space is ultra Hausdorff if and only if it is an R_{cl}-space. In Section 5 we discuss the relation between R_{cl}-supercontinuous functions and R_{cl}-spaces. Section 6 is devoted to the study of function spaces wherein it is shown that the function space of all $R_{cl}(X, Y)$ of all R_{cl}-supercontinuous functions from a topological space X into a uniform space Y is closed in Y^X in the topology of uniform convergence and the condition for its completeness is outlined.

2. Preliminaries and basic definitions

Let X be a topological space. A subset A of a space X is called ***regular*** G_δ***-set*** [23] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap\limits_{n=1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n^0$,where each F_n is a closed subset of X (here F_n^0 denotes the interior of F_n). The complement of a regular G_δ-set is called a ***regular*** F_σ***-set***. Any union of regular F_σ-sets is called d_δ***-open*** [17]. The complement of a d_δ-open set is referred to as a d_δ***-closed set***.

A subset A of a space X is said to be ***regular open*** if it is the interior of its closure, i.e., $A = \overline{A}^0$. The complement of a regular open set is referred to as a ***regular closed set***. Any union of regular open sets is called δ***-open set*** [40]. The complement of a δ-open set is referred to as a δ***-closed set***. Any intersection of closed G_δ-sets is called ***d-closed set*** [16]. Any intersection of zero sets is called ***z-closed set*** ([15, 30]).

A collection β of subsets of a space X is called an ***open complementary system*** [9] if β consists of open sets such that for every $B \in \beta$, there exist $B_1, B_2, \ldots \in \beta$ with $B = \cup\{X \setminus B_i : i \in N\}$. A subset A of a space X is called a ***strongly open*** $\boldsymbol{F_\sigma}$***-set*** [9] if there exists a countable open complementary system $\beta(A)$ with $A \in \beta(A)$. The complement of a strongly open F_σ-set is called ***strongly closed G_δ-set***. Any intersection of strongly closed G_δ-sets is called d^****-closed set*** [31].

Definition 2.1. A topological space X is said to be

(i) **functionally regular** ([3, 39]) if for each closed set F in X and each $x \notin F$ there exists a continuous real-valued function f defined on X such that $f(x) \notin \overline{f(F)}$.

(ii) **ultra Hausdorff** [35] if every pair of distinct points in X are contained in disjoint clopen sets.

(iii) R_z**-space** ([20, 33]) if for each open set U in X and each $x \in U$ there exists a z-closed set H containing x such that $H \subset U$; equivalently U is expressible as a union of z-closed sets.

(iv) R_δ**-space** [19] if for each open set U in X and each $x \in U$ there exists a δ-closed set H containing x such that $H \subset U$; equivalently U is expressible as a union of δ-closed sets.

(v) R_0**-space** ([5],[38][1] [28]) if for each open set U in X and each $x \in U$ implies that $\overline{\{x\}} \subset U$.

(vi) R_1**-space** ([42][2] [5]) if $x \notin \overline{\{y\}}$ implies that x and y are contained in disjoint open sets.

(vii) π_2**-space** [38][3] ($\equiv P_\Sigma$-space [41]$\equiv$ strongly s-regular space [7]) if every open set in X is expressible as a union of regular closed sets.

(viii) π_0**-space** ([38, p 98]) if every nonempty open set in X contains a nonempty closed set.

Definition 2.2 ([19]). A space X is said to be an

(i) R_{D_δ}**-space** if for each open set U in X and each $x \in U$ there exists a regular G_δ-set H containing x such that $H \subset U$; equivalently U is expressible as a union of regular G_δ-sets.

(ii) R_{d_δ}**-space** if for each open set U in X and each $x \in U$ there exists a d_δ-closed set H containing x such that $H \subset U$; equivalently U is expressible as a union of d_δ-closed sets.

(iii) R_D**-space** if for each open set U in X and each $x \in U$ there exists a closed G_δ-set H containing x such that $H \subset U$; equivalently U is expressible as a union of closed G_δ-sets.

(iv) R_d**-space** if for each open set U in X and each $x \in U$ there exists a d-closed set H containing x such that $H \subset U$; equivalently U is expressible as a union of d-closed sets.

[1]Vaidyanathswamy calls R_0-axiom as π_1-axiom in his text book (see [38, p 98]). Császár calls an R_0-space as S_1-space in [4].

[2]Yang [42] in his studies of paracompactness refers an R_1-space as a T_2-space. Császár calls an R_1-space as S_2-space in [4].

[3]π_2-spaces were defined by Vaidyanathswamy [38] (1960) and rediscovered by Wong [41] (1981) and Ganster [7] (1990) with different terminologies.

Definition 2.3 ([20])**.** A space X is said to be an

(i) R_{D^*}**-space** if for each open set U in X and each $x \in U$ there exists a strongly closed G_δ-set H containing x such that $H \subset U$; equivalently U is expressible as a union of strongly closed G_δ-sets.
(ii) R_{d^*}**-space** if for each open set U in X and each $x \in U$ there exists a d^*-closed set H containing x such that $H \subset U$; equivalently U is expressible as a union of d^*-closed sets.

Definition 2.4. A space X is said to be

(i) **D-completely regular** [9] if it has a base of strongly open F_σ-sets.
(ii) **D-regular** [9] if it has a base of open F_σ-sets.
(iii) **weakly regular** [9] if it has a base of F_σ-neighbourhoods.
(iv) D_δ**-completely regular** [18] if it has a base of regular F_σ-sets.

3. R_{cl}-spaces and hierarchy of seperation axioms

Definition 3.1. Let X be a topological space. Any intersection of clopen sets in X is called **cl-closed** [32]. An open subset U of X is said to be r_{cl}**-open** [37] if for each $x \in U$ there exists a cl-closed set H containing x such that $H \subset U$; equivalently U is expressible as a union of cl-closed sets.

Definition 3.2 ([37])**.** A topological space X is said to be an R_{cl}**-space** if every open set in X is rcl-open.

It is clear from the definitions that every zero dimensional space as well as every ultra Hausdorff space is an R_{cl}-space. The space of strong ultrafilter topology [36, Example 113, p.133] is a Hausdorff extremally disconnected R_{cl}-space which is not zero dimensional.
The comprehensive diagram (Figure 1) well reflects the place of R_{cl}-spaces in the hierarchy of separation axioms related to the theme of the present paper and certain other topological invariants and extends several existing diagrams in the literature (see [9, 18, 19]).

However, most of the implications of Figure 1 are irreversible (see [9, 18, 19, 20]). We reproduce the diagram (Figure 2) from [20] concerning separation axioms between functionally regular space and R_0-space, which is complementary to Figure 1.

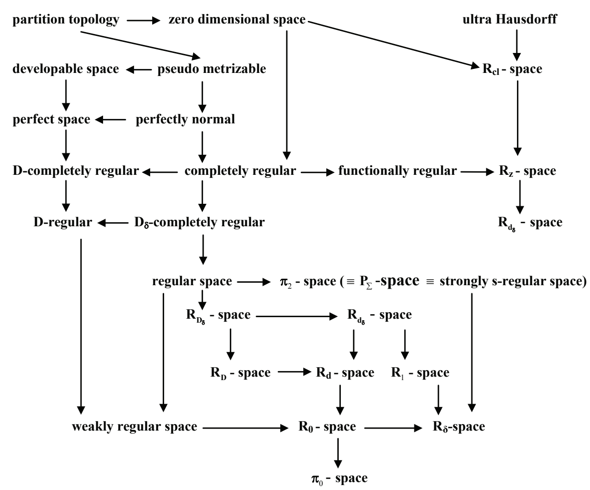

FIGURE 1.

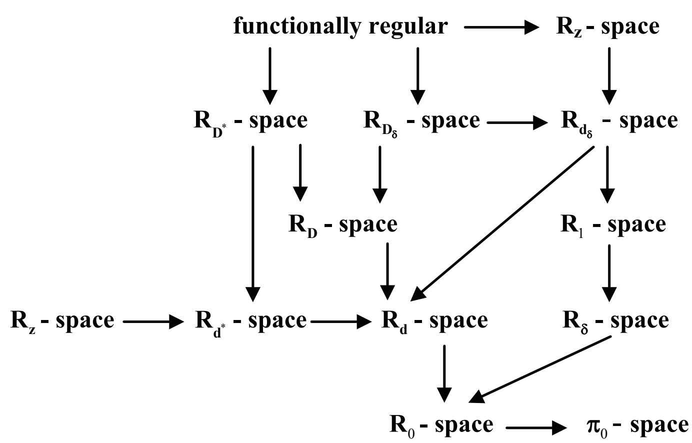

FIGURE 2.

4. Basic properties of R_{cl}-spaces

Definition 4.1. Let X be a topological space. A point $x \in X$ is said to be an r_{cl}**-adherent point** of a set $A \subset X$ if every r_{cl}-open set containing x intersects A. Let A_{rcl} denote the set of all r_{cl}-adherent points of the set A. Then $A \subset \overline{A} \subset A_{rcl}$. The set A is r_{cl}-closed if and only if $A = A_{rcl}$.

Lemma 4.2. *The correspondence $A \to A_{rcl}$ is a Kuratowski closure operator.*

Theorem 4.3. *Let X be a topological space. Consider the following statements:*

(i) *X is an R_{cl}-space*
(ii) *For each $x \in X$ and for each open set U containing x, $\{x\}_{rcl} \subset U$*
(iii) *There exists a subbase **S** for X such that $x \in S \in \boldsymbol{S} \Rightarrow \{x\}_{rcl} \subset S$*
(iv) $x \in \{y\}_{rcl} \Rightarrow y \in \{x\}_{rcl}$
(v) $x \in \{y\}_{rcl} \Rightarrow \{x\}_{rcl} = \{y\}_{rcl}$

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.

Proof. $(i) \Rightarrow (ii)$. Let $x \in X$ and let U be an open set containing x. Since X is an R_{cl}-space, there exists an r_{cl}-closed set A such that $x \in A \subset U$. Consequently $\{x\}_{rcl} \subset U$.
The assertions $(ii) \Rightarrow (i)$ and $(ii) \Leftrightarrow (iii)$ are trivial.
$iii) \Rightarrow (iv)$. Since every subbasic open set containing y contains $\{y\}_{rcl}$, every basic open set containing y contains $\{y\}_{rcl}$ and hence it contains x. So $y \in \{x\}_{rcl}$.
$(iv) \Rightarrow (v)$. Since $x \in \{y\}_{rcl}$, $y \in \{x\}_{rcl}$. So $x \in \{y\}_{rcl}$ and $y \in \{x\}_{rcl}$ implies $\{x\}_{rcl} \subset \{y\}_{rcl}$ and $\{y\}_{rcl} \subset \{x\}_{rcl}$, and hence $\{x\}_{rcl} = \{y\}_{rcl}$.
The implication $(v) \Rightarrow (iv)$ is obvious. □

Theorem 4.4. *For a topological space X the following statements are equivalent:*

(i) $\{x\}_{rcl} \neq \{y\}_{rcl}$ *implies that x and y are contained in disjoint open sets*
(ii) $x \notin \{y\}_{rcl}$ *implies that x and y are contained in disjoint open sets*
(iii) *A is compact set and $\{x\}_{rcl} \cap A = \varnothing$ implies x and A are contained in disjoint open sets*
(iv) *If A and B are compact sets, and $\{a\}_{rcl} \cap B = \varnothing$ for every $a \in A$, then A and B are contained in disjoint open sets.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $x \notin \{y\}_{rcl}$. Then $\{x\}_{rcl} \neq \{y\}_{rcl}$ and so by (i) x and y are contained in disjoint open sets.
$(ii) \Rightarrow (iii)$. Let A be a compact set and suppose that $\{x\}_{rcl} \cap A = \varnothing$. So for each $a \in A$, $a \notin \{x\}_{rcl}$ by (ii) there exist disjoint open sets U_a and V_a containing a and x, respectively. Thus the collection $\nu = \{U_a : a \in A\}$ is an open cover of the compact set A and so there exists a finite subcollection $\{U_{a_1}, ..., U_{a_n}\}$ of ν which covers A. Let $U = \cup_{i=1}^{n} U_{a_i}$ and $V = \cap_{i=1}^{n} V_{a_i}$. Then U and V are disjoint open sets containing A and x, respectively.
$(iii) \Rightarrow (iv)$. Suppose that A and B are compact and $\{a\}_{rcl} \cap B = \varnothing$ for every $a \in A$. Then by (iii) for each $a \in A$ there exist disjoint open sets U_a and V_a containing a and B, respectively. The collection $\nu = \{U_a : a \in A\}$ is an open cover of the compact set A and so there exists a finite subcollection $\{U_{a_1}, ..., U_{a_n}\}$ of ν which covers A. Let $U = \cup_{i=1}^{n} U_{a_i}$ and $V = \cap_{i=1}^{n} V_{a_i}$. Then U and V are disjoint open sets containing A and B, respectively.
$(iv) \Rightarrow (i)$. Suppose $\{x\}_{rcl} \neq \{y\}_{rcl}$. Then either $x \notin \{y\}_{rcl}$ or $y \notin \{x\}_{rcl}$. For definiteness assume that $y \notin \{x\}_{rcl}$. Then $\{x\}_{rcl} \cap \{y\}_{rcl} = \varnothing$ and so by (iv) there exist disjoint open sets U and V containing x and y, respectively.

□

Theorem 4.5. *The disjoint topological sum of any family of R_{cl}-spaces is an R_{cl}-space.*

Theorem 4.6. *The property of being an R_{cl}-space is closed under initial sources, i.e., the property of being an R_{cl}-space is an initial property.*

Proof. Let $\{f_\alpha : X \to Y_\alpha : \alpha \in \Lambda\}$ be a family of functions, where each Y_α is an R_{cl}-space and let X be equipped with initial topology. Let U be any open set in X and let $x \in U$. Then there exist $\alpha_1, ..., \alpha_n \in \Lambda$ and open sets $V_i \in Y_{\alpha_i}(i = 1, ..., n)$ such that $x \in f_{\alpha_1}^{-1}(V_1) \cap ... \cap f_{\alpha_n}^{-1}(V_n) \subset U$. Since each Y_α is an R_{cl}-space, there exists a cl-closed set A_{α_i} in $Y_{\alpha_i}(i = 1, ..., n)$ such that $f_{\alpha_i}(x) \in A_{\alpha_i} \subset V_i$. Since each f_α is continuous, it follows that each $f_{\alpha_i}^{-1}(A_{\alpha_i})$ is a cl-closed set in X. Let $A = \cap_{i=1}^{n} f_{\alpha_i}^{-1}(A_{\alpha_i})$. Since any intersection of cl-closed sets is a cl-closed, A is a cl-closed set in X and $x \in A \subset U$ so X is an R_{cl}-space. □

As an immediate consequence of Theorem 4.6 we have the following.

Theorem 4.7. *The property of being an R_{cl}-space is hereditary, productive, sup-invariant, preimage invariant and projective*[4].

Theorem 4.8. *The category of R_{cl}-spaces and continuous maps is a full isomorphism closed monoreflective as well as epireflective subcategory of TOP*[5].

The following result gives a factorization of ultra Hausdorff property with R_{cl}-space as an essential ingredient.

Theorem 4.9. *Every ultra Hausdorff space is an R_{cl}-space. Conversely, every T_0, R_{cl}-space is an ultra Hausdorff space.*

Proof. The first assertion is immediate, because in this case every singleton is cl-closed and so every open set is the union of cl-closed sets. Conversely, suppose that X is a T_0, R_{cl}-space and let $x, y \in X$, $x \neq y$. By T_0-property of X there exists an open set U containing one of the points x and y but not both. To be precise, assume that $x \in U$. Since X is an R_{cl}-space, there exists a cl-closed set A such that $x \in A \subset U$. Let $A = \cap\{C_\alpha : \alpha \in \Lambda\}$, where each C_α is a clopen set. Then there exists an $\alpha_0 \in \Lambda$ such that $y \notin C_{\alpha_0}$. Hence C_{α_0} and $X \setminus C_{\alpha_0}$ are disjoint clopen sets containing x and y, respectively and so X is an ultra Hausdorff space. □

5. R_{cl}-supercontinuous functions and R_{cl}-spaces

Definition 5.1 ([37])**.** A function $f : X \to Y$ from a topological space X into a topological space Y is said to be R_{cl}**-supercontinuous** if for each $x \in X$ and for each open set V containing f(x), there exists an r_{cl} open set U containing x such that $f(U) \subset V$.

[4] A topological property P is said to be projective if whenever a product space has property P every co-ordinate space possesses property P.

[5] For the definition of categorical terms we refer the reader to Herrlich and Strecker [11].

It is immediate from the definition that every continuous function defined on an R_{cl}-space is R_{cl}-supercontinuous.
Next we quote the following result from [37].

Theorem 5.2 ([37, Theorem 4.11]). *Let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))_{\alpha\in\Lambda}$, where $f_\alpha : X \to X_\alpha$ is a function for each $\alpha \in \Lambda$. Let $\prod_{\alpha\in\Lambda} X_\alpha$ be endowed with the product topology. Then f is R_{cl}-supercontinuous if and only if each f_α is R_{cl}-supercontinuous.*

Now we give an alternative short proof of the following result from [37].

Theorem 5.3 ([37, Theorem 4.13]). *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is R_{cl}-supercontinuous if and only if f is R_{cl}-supercontinuous and X is an R_{cl}-space.*

Proof. Observe that $g = 1_X \times f$, where 1_X denotes the identity function defined on X. Now by Theorem 5.2, g is R_{cl}-supercontinuous if and only if 1_X and f both are R_{cl}-supercontinuous. Again 1_X is R_{cl}-supercontinuous implies that each open set in X is r_{cl}-open and so X is an R_{cl}-space. □

Theorem 5.4. *Let $f : X \to Y$ be an R_{cl}-supercontinuous open bijection. If either of the space X and Y is a T_0-space, then X and Y are homeomorphic ultra Hausdorff spaces.*

Proof. By [37, Theorem 5.1] X and Y are homeomorphic R_{cl}-spaces. The last part of the theorem is immediate in view of the fact that a T_0, R_{cl}-space is ultra Hausdorff (Theorem 4.9). □

6. Function spaces

It is a well known fact that the function space $C(X,Y)$ of all continuous functions from a topological space X into a uniform space Y is not necessarily closed in Y^X in the topology of pointwise convergence. However, it is closed in Y^X in the topology of uniform convergence. It is of fundamental importance in topology, analysis and several other branches of mathematics and its applications to know whether a given function space is closed / compact / complete in Y^X or $C(X,Y)$ in the topology of pointwise convergence / uniform convergence. Results of this nature and Ascoli type theorems abound in the literature (see [1, 12]). Sierpinski [29] showed that the set of all connected (Darboux) functions from a topological space X into a uniform space Y is not necessarily closed in Y^X in the topology of uniform convergence. In contrast, Naimpally [25] showed that the set of all connectivity functions from a space X into a uniform space Y is closed in Y^X in the topology of uniform convergence. Moreover, in [26] Naimpally introduced the notion of graph topology Γ for a function space and proved that the set of all almost continuous functions in the sense of Stalling [34] is not only closed in Y^X in the graph topology but it represents the closure of $C(X,Y)$ in the graph topology. In the same vein, Hoyle [10] showed that the set SW(X, Y) of all somewhat continuous functions from a space X into a uniform space Y is closed in Y^X in the topology of uniform convergence. Furthermore, Kohli and Aggarwal in [14] proved that the function space $SC(X,Y)$ of quasicontinuous ($\equiv$ semicontinuous) functions, $C_\alpha(X,Y)$ of α-continuous functions, and L(X, Y) of cl-supercontinuous functions are closed in Y^X in the topology of uniform convergence. In this section we strengthen the results of [14] and show that the set $R_{cl}(X,Y) \supset L(X,Y)$ of all R_{cl}-supercontinuous functions is closed in Y^X in the topology of uniform convergence.

Definition 6.1. A subset A of a topological space X is said to be

(i) **semi open** [22] ($\equiv$ **quasi open** [13]) if there exists an open set U in X such that $U \subset A \subset \overline{U}$
(ii) α**-open** [27] if $A \subset \overline{(A^0)}^0$
(iii) **cl-open** [32] if for each $x \in A$ there exists a clopen set H such that $x \in H \subset A$.

Definition 6.2. A function $f : X \to Y$ from a topological space X into a topological space Y is said to be a

(i) **connected (Darboux)** function if $f(A)$ is connected for every connected set $A \subset X$
(ii) **connectivity** function if the graph of every connected subset of X is a connected subset of $X \times Y$
(iii) **semicontinuous** [22] (**quasicontinuous** [13]) if $f^{-1}(V)$ is semi open in X for every open set V in Y
(iv) α**-continuous** [24] if $f^{-1}(V)$ is α-open in X for every open set V in Y
(v) **somewhat continuous** [8] if for each open set V in Y such that $f^{-1}(V) \neq \varnothing$, then there exists a nonempty open set U in X such that $U \subset f^{-1}(V)$, i.e. $(f^{-1}(V))^0 \neq \varnothing$.

Remark 6.3. Somewhat continuous functions have also been referred to as feebly continuous (see [2, 6]) in the literature. However, Frolik [6] requires functions to be onto.

We now recall the notion of the topology of uniform convergence. Let $Y^X = \{f : X \to Y \text{ is a function}\}$ be the set of all functions from a topological space X into a uniform space (Y,ν), where ν is a uniformity on Y. Let $F \subset Y^X$. A basis for the uniformity of uniform convergence u for F is the collection $\{W(V) : V \in \nu\}$, where $W(V) = \{(f,g) \in F \times F : (f(x), g(x)) \in V \text{for all x} \in X\}$. The uniform topology associated with u is called the topology of uniform convergence. For details we refer the reader to [12].

Definition 6.4 ([12])**.** A uniform space (Y,ν) is said to be **complete** if and only if every Cauchy net in Y converges to a point in Y.

Theorem 6.5 ([12, p. 194])**.** *A product of uniform spaces is complete if and only if each co-ordinate space is complete.*

Theorem 6.6. *Let X be a topological space and let (Y, ν) be a uniform space. Then the set $R_{cl}(X, Y)$ of all R_{cl}-supercontinuous functions from X into Y is closed in Y^X in the topology of uniform convergence. Further, if Y is a complete uniform space, then so is the function space $R_{cl}(X, Y)$ in the topology of uniform convergence.*

Proof. Let $f \in Y^X$ be the limit point of $R_{cl}(X, Y)$ which is not R_{cl}-supercontinuous at $x \in X$. Then there exists $V \in \nu$ such that $f^{-1}(V[f(x)])$ does not contain any r_{cl}-open set containing x. Choose a symmetric member W of ν such that $WoWoW \subset V$. Since f is a limit point of $R_{cl}(X, Y)$, there exists $g \in R_{cl}(X, Y)$ such that $g(y) \in W[f(y)]$ for all $y \in X$. Then $g \subset Wof$ and $g^{-1} \subset f^{-1}oW^{-1} = f^{-1}oW$ and hence $g^{-1}oWog \subset f^{-1}oWoWoWof \subset f^{-1}oVof$. Therefore $g^{-1}[W(g(x))] \subset f^{-1}(V[f(x)])$. Since $f^{-1}(V[f(x)])$ does not contain any r_{cl}-open set containing x, neither does $g^{-1}[W(g(x))]$ which contradicts R_{cl}-supercontinuity of g. Therefore $f \in R_{cl}(X, Y)$. The last assertion is immediate in view of Theorem 6.5 and the fact that a closed subspace of complete uniform space is complete. □

Remark 6.7. In view of the above discussion we extend the following inclusions diagram from [14].
$L(X, Y) \subset R_{cl}(X, Y) \subset C(X, Y) \subset C_\alpha(X, Y) \subset SC(X, Y) \subset SW(X, Y) \subset Y^X$.
Since in the topology of uniform convergence each of the above function space is a closed subspace of its succeeding one, the completeness of any one of them implies that of its predecessor. In particular, if Y is complete, then each of the above function space is complete.

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